

**SURMOUNTING THE CARTESIAN CUT THROUGH PHILOSOPHY,
PHYSICS, LOGIC, CYBERNETICS, AND GEOMETRY:
SELF-REFERENCE, TORSION, THE KLEIN BOTTLE, THE TIME
OPERATOR, MULTIVALUED LOGICS AND QUANTUM MECHANICS**

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Abstract: In this transdisciplinary article which stems from philosophical considerations (that depart from phenomenology -after Merleau-Ponty, Heidegger and Rosen- and Hegelian dialectics), we develop a conception based on topological (the Moebius surface and the Klein bottle) and geometrical considerations (based on torsion and non-orientability of manifolds), and multivalued logics which we develop into a unified world conception that surmounts the Cartesian cut and Aristotelian logic. The role of torsion appears in a self-referential construction of space and time, which will be further related to the commutator of the True and False operators of matrix logic, still with a quantum superposed state related to a Moebius surface, and as the physical field at the basis of Spencer-Brown's primitive distinction in the protologic of the calculus of distinction. In this setting, paradox, self-reference, depth, time and space, higher-order non-dual logic, perception, spin and a time operator, the Klein bottle, hypernumbers due to Musès which include non-trivial square roots of ± 1 and in particular non-trivial nilpotents, quantum field operators, the transformation of cognition to spin for two-state quantum systems, are found to be keenly interwoven in a world conception compatible with the philosophical approach taken for basis of this article. The Klein bottle is found not only to be the topological information for self-reference and paradox whose logical counterpart in the calculus of indications are the paradoxical imaginary time waves, but also a classical-quantum transformer (Hadamard's gate in quantum computation) which is indispensable to be able to obtain a complete multivalued logical system, and still to generate the matrix extension of classical connective Boolean logic. We further find that the multivalued logic that stems from considering

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the paradoxical equation in the calculus of distinctions, and in particular, the imaginary solutions to this equation, generates the matrix logic which supersedes the classical logic of connectives and which has for particular sub-theories fuzzy and quantum logics. Thus, from a primitive distinction in the vacuum plane and the axioms of the calculus of distinction, we can derive by incorporating paradox, the world conception succinctly described above.

KEYWORDS. Time operator; time-waves; self-reference; torsion geometry; quantum mechanics; quantum computation; spin; radical recursion; Muses hypernumbers; nilpotents; neurology, Klein bottle; Cartesian cut; calculus of distinctions; multivalued logics; matrix logics; philosophical phenomenology; cognition; perception; eikonal equation; photon; cybernetics; Fibonacci sequence; systems theory; semiotics; endophysics; implicate and explicate orders; holomoment; mind-matter problem; meta-algorithmic level.

1 Introduction

There are two current views in physics which seem irreconcilable, the geometrical approach that was framed in General Relativity in terms of geometries which have for basic element a metric and which points to a reality which is purported to be independent of the subject, and Quantum Mechanics, which in whatever interpretation is taken, it indicates the essential singularity in the photon and quantization (and is deeply manifested already in the Planck scale for which space and time loses its consistence), and which points to the subject's participation in the observed world. This separation seems to divide the scientific community in a way beyond the possibility of reaching a common ground. In the other hand, the naissance of the newest branch of Science, Cybernetics, which had the fortune of being the child of transdisciplinary contributions from its very beginning (mathematician: Wiener, anthropologists: Bateson and Mead, neuroscientists: MacCulloch, physicists: von Foerster, philosophers: Gunther, and other prominent thinkers) dealt with representations of the 'interior' world and its technological implementations, and in doing this, was obliged to reach an understanding of these representations and of what they stood for.

We quote here some definitions [1]: "Cybernetics could be thought of as a recently developed science, although to some extent it cuts across existing sciences. If we think of Physics, Chemistry, Biology, etc. as traditional sci-

ences, then Cybernetics is a classification which cuts across them all. (F.H. George) ."A branch of mathematics dealing with problems of control, recursiveness, and information, ...the study of form and pattern,...the biggest bite out of the fruit of the Tree of Knowledge that mankind has taken in the last 2000 years." (G. Bateson). "A science concerned with the study of systems of any nature which are capable of receiving, storing, and processing information so as to use it for control" (Kolmogorov) "Should one name one central concept, a first principle, of cybernetics, it would be circularity" (Von Forster). "At last there is a unifying framework that suspends long-held differences between science and art, and between external reality and internal belief." (Paul Pangaro). "Here is something philosophically or theoretically pregnant about cybernetics. There is a kind of seductive mystery or glamour that attaches to it. And the origin of this, I think, is that cybernetics is an instantiation of a different paradigm from the one in which most of us grew up, the reductive, linear, Newtonian, paradigm that still characterises most academic work in the natural and social sciences (and engineering and the humanities, too)-the classical sciences, as Ilya Prigogine and Isabelle Stengers call them. ... It appears to me, though, that historians have yet to get seriously to grips with this aspect of cybernetics. (Andrew Pickering). "Cybernetics was originally formulated as a way of producing mathematical descriptions of systems and machines. It solved the paradox of how fictional goals can have real-world effects by showing that information alone (detectable differences) can bring order to systems when that information is in a feedback relation with that system. This essentially bootstraps perception (detection of differences) into purpose". "The theory of interconnectedness of possible dynamic self-regulated systems with their subsystems" (G. Klaus). "First order cybernetics: The cybernetics of observed systems. Second order cybernetics: The cybernetics of observing systems." (F. Varela)

In these definitions there are several themes that recur: self-organization, recursivity (or circularity), forms and patterns, the action of the subject over the object (a system), an inner world (of machines or systems in general) which is to be steered (that is the Greek etymology of cybernetics), the problem of the nature and management of information (we recall here Wheeler's "it from bit"), the possibility of overcoming the current paradigms, the mathematical description of all this, and further a second-order perspective which is that of Varela: the cybernetics of cybernetics. In this perspec-

tive, the steerer ² has to be included in the system that is steered, and thus self-reference is placed as the lifeworld of the system in which the observer cannot be excluded any longer. Thus, paradoxically in the intent of producing autonomous mechanized systems, which would operate by programming and by the action of switches, latches or controlling gauges (anholonomic variables, as called by physicist and cybernetist W. Pattee [55]), controlling variables that cannot be separated from the system that they control were encountered, and thus we find clearly a second-order cybernetics in which the steerer (embodied in the anholonomic variables) cannot be detached from the system under control, and thus subject and object become fused into a single being from which the subject has projected himself through the non-holonomic variables. As Pattee puts it referring to the anholonomic variables: "they bridge all epistemic cuts, between the controller and the controlled, the classifier and the classified, the observer and the observed."³

²Trans-architect founder and theorist, Marcos Novak, whose work deals with cybernetics, self-reference and architecture, has claimed that 'kybernetes' has a more ancient root than the one taken by the founders of Cybernetics, related to flying rather than steering.

³With relation to semiotics, life, and computers, Pattee further observes that " We have come to think of symbol systems as having no relation to physical laws. This apparent independence of symbols and physical laws is a characteristic of all highly evolved languages, whether natural or formal. They have evolved so far from the origin of life and the genetic symbol systems that the practice and study of semiotics do not appear to have any necessary relation whatsoever to physical laws." In his understanding, one of the paradigmatic examples of this fiction is the programmable computer, which "for the user, the computer function can be operationally described as a physics-free machine, or alternatively as a "symbolically controlled, rule-based (syntactic) machine. Its behavior is usually interpreted as manipulating meaningful symbols, but that is another issue. The computer is a prime example of how the apparently physics-free function or manipulation of memory- based discrete symbol systems can easily give the illusion of strict isolation from physical dynamics. This illusion of isolation of symbols from matter can also arise from the apparent arbitrariness of the epistemic cut. It is the essential function of a symbol to stand for something its referent that is, by definition, on the other side of the cut. This necessary distinction that appears to isolate symbol systems from the physical laws governing matter and energy allows us to imagine geometric and mathematical structures, as well as physical structures and even life itself, as abstract relations and Platonic forms. I believe this is the conceptual basis of Cartesian mind-matter dualism. This apparent isolation of symbolic expression from physics is born of an epistemic necessity, but ontologically, it is still an illusion. In other words, making a clear distinction is not the same as isolation from all relations. We clearly separate the genotype from the phenotype, but we certainly do not think of them as isolated or independent of each other.

Thus in this latest newborn of Science we have found what physics was unable to get its grips to: the fusion of object with subject (the observer, as physics has called it though in a partial stance, which is the subject qua object), which appears in two forms in cybernetics: self-reference and anholonomic non-integrable variables. The former is related to consciousness (and consequently to the so-called mind-matter problem), the latter finds its modelization in mathematics and physics through non-integrable constraints and more generally through anholonomic tetrads (on studying gravitation on four-dimensional spacetime, or more generally, n -beins on n -dimensional spacetime manifolds), which produce non-null torsion due to its non-integrability [23]. In distinction with General Relativity, where tetrads are integrable and thus has null torsion, here we can see where Einstein with General Relativity standed in terms of constructing a view of the world which incorporated the epistemic cut (also called Cartesian, in acknowledgement of its brainfather, Descartes) through the geometrical representation that was chosen, purely metric and thus torsionless [23].

It is the objective of this article to present a philosophic contemplation in relation with this Cartesian cut ⁴ and the mathematical and physical construction of what we propose to be a solution that overcomes this gap, to

These necessary non-integrable equations of constraint that bridge the epistemic cut and thereby allow for memory, measurement, and control are on the same formal footing as the physical equations of motion. They are called non-integrable precisely because they cannot be solved or integrated independently of the law-based dynamics. Consequently, the idea that we could usefully study life without regard to the natural physical requirements that allow effective symbolic control is to miss the essential problem of life: how symbolic structures control dynamics⁷. Returning to the hidden structure of a sign argued by Pattee, the following example is illustrative. We cut the word information with a sign, -, to produce in-formation; the meaning of the former is about a transmission from an emitter to a receiver, which decodes its meanings initiating the hermeneutical (interpretative) process but is still a detached observer, while the meaning of 'in-formation' is that of a causative field that creates the information, an implicate order field in the sense of D. Bohm in which message and subject participate in an holomovement that sustains both in an integral structure [7]. It is clear then that in-formation is physical (yet, more properly, holographic. We shall reencounter this on dealing with matrix logic below.) Further more elaborate examples are provided by the 'alephbet', in the work of Stan Tenen further elaborated by Jay Kappraff, that manifests the complex physical, mathematical, cognitive and hermeneutical structures implicit to the Hebrew letters [78].

⁴For a critique and a perspective from a neurophysiologist and cognitive scientist, the work of Damasio is relevant [10].

find a new understanding of the relation between the 'external' and 'internal' worlds, which we shall elaborate in terms of self-reference, torsion and an original distinction which in the sense of Pattee is the physical embodiment of torsion as a field of non-anholonomy which is connected to the subject itself and the geometrical world created by this logo-physical (for reasons to be elaborated below) field, but which we shall see appearing in this original distinction which in the act of abstracting it as a physical independent sign will yield a most complex logic, which is related to quantum physics for two-state systems, and quantum field theory operators, which can be represented by nilpotent hypernumbers, which are related to logical operators; nilpotence has become a central issue in a universal semantic rewrite system for physics due to Rowlands [66].

2 The Cartesian Cut, Aristotelian Dualism, the Classical Model of Space and the Trivialization of Time

In Aristotelian thought from which stemmed the Western scientific tradition we have as its backbone a dualist conception which expresses itself in the two-valued logic and the principle of non-contradiction. According to philosopher and cybernetist G. Gunther, this dualism is expressed in the exhaustive division in the characterization of the universe with two values. One value is about designation and thus encompasses all what the universe is and what can be said of it [17]. The other value is non-designative and due to the completeness of the designative value, this non-designative value points to no ontological object or phenomenon. This eliminates subjectivity from the universe since subjectivity is incorporated into the subject qua object and the discourse of it falls into the designative value. This is best characterized by Shakespeare dictum: "To be or not to be". Here being is a static condition of an object placed in space in the Cartesian mindset of object-in-space-before-subject. Greek thinkers interrogated on the nature of time as belonging to the objective designative value or to the non-designative subjective one. The response of the Eleatic school whose most prominent figure was Parmenides was that time does not exist, the universe is static and the subject is part of the universe. Heraclitus response was the non self-identity of beings in

sequential time, or formalized in the laws of thought, A is not A at different sequential times. So here although the principle of contradiction is violated this is in a certain ordering of time which is linear and exterior to being. Newton maintained this exterior universal character of time while keeping the principle of non-contradiction of being, and keeping as well the Cartesian mindset of object-in-space-before-subject, where the subject is itself a static object with respect to whom beings manifest themselves. Einstein gave one step ahead of Newton by placing time in the same ontological position of space by identifying them as purely objective characters of the now promoted blend denominated spacetime. But now to preserve invariance of the knowledge of objects under examination the subjects qua objects must be related by certain transformation rules which are the designed by the Lorentz group in Special Relativity and the general group of invertible spacetime transformations (diffeomorphisms) in the case of General Relativity. So in Einstein's approach subjectivity as cognition is incorporated into the theory as these rules of transformations and the system of coordinates. Consistency of cognition requires, according to Einstein, that 'exterior' spacetime be continuous and thus singularities cannot be part of the universe, starting by the subject which is in this conception to be reckoned as an unacknowledged singularity.

Returning to our discussion on time, it is in Hegel's system that a new approach that relates it to subjectivity: Time is related to the laws of thought [22]. This laws are of internal necessity, so while they correspond to the designative value now they point to the internal world of the subject. In Hegel's dialectics we have a fusion of the designative value of objects and the non-designative value of subjectivity. Gunther's analysis of the Hegelian stance is that the elimination of time is related to the isomorphic character of the two-valued logic based on the principle of Tertium-non-datur (no third value): p and not p is false, which establishes the duality between conjunction and disjunction. This isomorphic character divides all objects which are the realm of the designative value into two classes: 1) Ortho-objects which can be conceived separately from any other object (ideally speaking; this establishes this separation in terms of the hidden assumption of the validity of the principle of non-contradiction: no A is not A) and 2) pseudo-objects which can be only be conceived with reference to other objects, their duals (e.g., right-left, night-day, etc., which is also dependent for its definition in the principle of course is also based in the assumption of two-valued logic where no merger of dualities can be the case, in opposition to Hegelian di-

alectics). Thus, the Zeno Paradox appears as a consequence of the impossible effort of reconciling the phenomena of change and motion with the static phenomenology of Being. In the Aristotelian tradition, Being is the class of all ortho-objects and thus designated by the single (designative) value, while time is a pseudo-object designated by three states (past, present and future), so the claimed isomorphism is non-existent. Thus in the Aristotelian tradition it is hard to find an ontological place for time, if there is any at all. This might be at the root of the trivialization of time that has been incorporated into science and human affairs at large with non-trivial consequences [59]. If instead we follow Gunther -momentarily- in abandoning the association of the notion of value in logic as fundamental to replace it by the notion of ontological locus with the elimination of Tertium-non-datur ⁵, we introduce the following ontological loci: 1) Being, 2) Its reflection on thought as an image (which requires implicitly the existence of light to form the image in terms of distinctions of light intensity or more primitively the existence of a boundary to establish images in terms of topology and this also requires light for establishing the boundary as a frontier between the inside of the object reflected and the outside); 3) Time . Thus 1) , 2) and 3) form the Ego complex. We can extend this to include 4) Thought as a process, which in the Aristotelian tradition is confused with thought as an image, and thus with the introduction of this fourth ontological locus we have internalized time as the difference that makes a difference in the sense of Bateson [6] ⁶, and finally the ontological locus 5) which operates as the detachment from the locus 4 (implying also locus 2)). Operationally speaking, locus 5) is an iteration or repetition of 1) to 4), so no infinite regression is possible. So a 4-loci logic is sufficient to describe Time, Being, and the Subject as the bearer of reflection of Being in thought and thought as a process. We shall later see how a 4-valued logic incorporates time as an imaginary locus of self-reference and the manifestation of paradox, so that this will flesh out how to further

⁵In fact, the present approach will upgrade the shallow non-self-referential notion of value which is in fact devaluated by its non-self-referential character in Aristotelian dualism. Indeed, the distinction true/false will appear to be secondary to the true process of self-reference from which they appear. In Hegel's dialectics the self-referential content of value is essential to his system; see pages 301, 557,[20]; we thank S. Johansen for these references.

⁶An important development that departed from Bateson's work is a differential epistemology which may be applied to both the 'exact' and 'human' sciences elaborated by S. Johansen [35].

the project of abandonment of the positivistic classical formula for time and as we shall also see, the classical formula of Cartesian space as well. Here imaginary means the process of extension of the number field of the reals to the hypercomplex numbers which in the Cartesian tradition do not appear as objectification of space nor time and more generally, the inclusion of hypernumbers which includes in addition to the non-trivial square roots of -1 , also non-trivial square roots of $+1$ [49]. As such, they are not included in the objective designative valorization of the world of *res extensa*, and thus lie in the side of the subject, not on the bridge, in the formula for the constitution of the world. ⁷

2.1 Perception, Cognition And The Phenomenological Philosophy of Space and Time

Philosopher Steven M. Rosen elaborated in terms of the phenomenological philosophic works of M. Merleau Ponty and M. Heidegger a theory of space and time which is ultimately related to the process of individuation of the subject [63]. In Merleau Ponty we find a critique of the classical formula object-before-subject-in-space [48]. Here all points are exterior one to the other. Rosen argues that "...classical space is the sheer boundedness serving as the medium for the unbounded subjects' operations upon bounded objects. The essential principle here is that of external relationship. That is, objects are simply external to each other appear within a spatial context of sheer externality (the "outside-of-one-another" of the multiplicity of points of Heidegger, p. 481, [21]. Further: "in the underlying lifeworld there is no object with boundaries so sharply defined that it is closed off completely from other objects."⁸ The lifeworld is characterized instead by the transpermeation of objects, by their mutual interpenetration, by the reciprocal insertion and intertwining of one in the other", as Merleau Ponty put it (p. 138, 1968) [48]. The key point is that if objects are related by mutual containence, no

⁷While multivalued logic had to wait in the Western world for the work of E. Post and Lujasiewicz in the XXth century [57], the Hindu philosopher Nagarjuna (150-250 C.E.) developed a 4-valued logic which is part of the Tibetan Buddhist tradition [51]. His system includes the loci: true, false, nor true nor false, and finally, true and false. We shall retrieve Nagarjuna's proposal further below in the setting of the present article.

⁸This fuzziness of the boundaries is the basis of what became in the last decades the so-called *fuzzy logics*.

separate container is required to mediate their relations, as would have been to be the case with externally related objects. They don't interpenetrate and the interactions are through forces, which in Einstein's General Relativity forces are substituted by the curvature of the geometry which still keeps the classical formula. Rosen conceives the world as a living object in which reality is a joint construction of the dialectical relation of object with subject; in the lifeworld space and time are not mere containers but the essential core of this relation. This idea of a participatory lifeworld is also implicit in the philosophy of B. Spinoza [75], and the expression of this participation is the *conatum*, by which the preservation of the subject vis-à-vis the exterior world, is due to the joint constitution of both in the lifeworld. It is important to mention here that to Merleau Ponty, the Cartesian formula is one of the absolute positivity of space: it is the absolute explicit openness, the sheer positive extension that constitutes the field of strictly external relations wherein unambiguous measurements can be made. In Merleau Ponty depth is the primordial dimension, a protodimension from which all Cartesian dimensions appear which are furthermore idealizations of it. Depth is a *self-containing* dimension, not merely a container for objects separated from it. Rosen: "Depth blends subject with object concretely, rather than serving as a static arena for the objectivation of the detached subject. The subject is an aspect of the indivisible cycle of action (a process) in which the contained and the uncontained, object and subject are "fleshed" and blended". This makes the object and the subject to have common properties as part of a fusion which constitutes them solidarily, so that being the case that the subject is extended qua object, it necessarily shares this property with objects, which can only be pointlike if we ascribe to points not the idealized structure we usually give them, but an extended structure, less we should be of need of abandoning altogether the concept of a pointlike structure which is the unextended projection of the *res cogitans* of the subject, and thus a purely idealized structure, and as such a reduction of the lifeworld of the fusion of subject with object. With respect to time, in Heidegger we find the notion of time-space or true time. Succinctly, for Heidegger time is not pointlike but rather extended nonlinearly as a relation between past (which denied to us by the linear-time-present is felt as memory), present (which in physiology is quantized as an interval of 50 msec [15]) and future which it is withheld from us yet is active (Libet's experiments [43]). Time extends nonlinearly and the three states of time are interwoven. Furthermore,

for Heidegger time is the protodimension and thus identified with the Merleau Ponty dimension of depth, and is the source for space. The nonlinear form of time is to Heidegger, the precondition for space. Nature appears to provide examples of this conception in the mathematical representation of the growth of the structure of conchoids as described in the seminal work in biology by Chris Illert, who proposed a mathematical model based in a simple two-parameter dependent algorithm which is able to reproduce the growth of all conchoids. In this model, at each point of development of a subclass of all conchoids, the self-branching ones, we find a decomposition of its evolution as a curve representing the development from past to present, an action forward in time through future time and a reversed back action through future time (the so called Gaitlin propagator) [32]. In the work of mathematician and philosopher Charles Musès we have a similar approach. Time is a protodimension, in fact with a negative dimension acting as an operator on each space dimension. Furthermore, in conceiving this protodimension Musès proposed that it is multidimensional and that a topology of time appears. There is a chronotopology in Nature through which time acts as an operator [50].

Now we would like to point out to one particular example in which the relation between depth and time can be understood as superposed. This is the case of the Necker cube (for images we refer to [52]), in fact an example taken by Merleau Ponty in his *Phenomenology of Perception*. Here we see that this cube drawn in the plane allows two perceptual interpretations which arise due to the lack of *privileged depth cues*, an issue of great importance for establishing vision [19,33,46], as we shall argue further below in general the need for cues to establish vision altogether. So which is the one viewed by the subject which if waiting for some time to elapse will see the other possible view, belies the Cartesian cut as explained by Rosen (page 173, 2004)[63], "this means that the speaking and thinking subject -no less than the sensing subject- is an embodied participant in the earthly transactions of the phenomenological lifeworld, not just a detached cogito". As Rosen further argues, in Merleau Ponty's analysis there is no full fusion of subject with object in terms of what the subject can experience as the perceptual flow is in the former author established from the subject to the object, denying thus a full self-referential feedback and exchange. This Cartesian cut maintained by this incomplete flow was also conceived in the studies of another psychological researcher mentioned by Rosen, A. Iran-Nejad, who claimed that "To

the extent that two different schemas must share the same knowledge components they cannot ... be held in mind simultaneously just as two meetings cannot be held at the same time to the extent that the same individuals must participate at both of them , p. 131 [34]. Rosen then observes that this claim is tantamount to the principle of non-contradiction, which further stands in distinction of the dialectical thought and quoting philosopher P. Angeles, " a logic of becoming that attempts to present the ever-changing processes of things" , (page 153)[2]. Further, Angeles: "Here contradiction exists in reality. It is impossible for the selfsame thing to be and not to be. Since there would be no contradiction in saying that A becomes not-A within the passage of time (our comment, Heraclitus view of time), dialectically it is understood that A is not-A at the time that it is A". Yet, this profession of Aristotelianism belies the structure of the quantum vacuum. Rosen: "the Apeironic Being as a dialectical life processes, certainly does not conform to Aristotelian structures. We know that Being's temporality is not the mere sequence of point-like structures, extensionless 'nows' ontically posited by Aristotle. It is only if time were Aristotelian that A and not-A could not be in the same instant. But the true time adumbrated by Heidegger possesses the dialectical thickness in which the future, past and present... belong together in the way that they offer themselves to one another" (Heidegger, page 14, [21]). Thus in this temporality of Being it is indeed possible for opposing perspectives of the cube to coincide 'simultaneously' (see below for further qualification of this simultaneity)" (page 176, 2004)[63]. Continuing with the qualifications on 'simultaneity' Rosen proceeds "I am proposing that we can apprehend the cube in such a way that its differing viewpoints overlap in time as well as in space ⁹... But there is a coincidence in the integrative way of viewing the cube, for perspectives are not related in a simple Aristotelian succession (first one, then the other)...instead the relation is one of internal mediation, of the mutual permeation are grasped as flowing through each other in a manner that blends space and time so completely that they are no longer recognizable in their familiar dichotomized forms. " (page 179, 2004)[63]. On further commenting the 'impossible figures' in Necker cubes [52], Rosen further comments "Its non-simultaneous aspect makes it clear that the perspectival integration of the cube does not negate the distinction

⁹We recall that physiologically, time is quantized to 50 msec (our quote)[15]. For more recent research on time windows in perception, see E. Ruhnau [15].

between sides. Faces of the cube are inside, and yet outside as well. So the feature of the separateness is not lost; rather, a unity is gained that is deeper than that of the simple dipolar Necker cube; without the mere eliminating opposition, the interpenetration of opposites is embodied. To be sure this is a paradox, and in concretely expressing it the act of perception performed with the cube, it is fleshed out more than words alone can do... Through this surface of paradox, one may symbolically gain a palpable glimmer of how subject and object can be opposed, as in reflective consciousness, and also, prereflectively one and the same” (page 180, 2004) [63]. We can find in the presentation of Rosen the same four ontological elements that Gunther proposed as the loci for a superation of Aristotelian logic as we already discussed at the beginning of this article: Being, thought as a reflection of Being, time, and thought as a process, now here integrated in the description of the surface of paradox which is the Necker cube.

Yet we would like to proceed further in this description of how depth comes to be perceived by the subject.¹⁰ Several cues are used for the formation of the perception of depth, such as occlusion, rotation of objects (so a perceptual spin is relevant to the formation of depth perception, an issue which will appear later in matrix logic and the transformation of the mean of

¹⁰Our insistence on an epistemology based on the fusion of subject and object, focusing in particular, for later integration, the mind processes and perception, is an attempt for coherence and consistence. A similar view with regards to establish an *Endophysics*-in the sense of Rossler- conception of the world in terms of perception, is also the ground of the work in the projective geometry of perceptual time of consciousness 'altered' states developed by M. Saniga [69]. Saniga's work is further related to the perspective first proposed by D. Finkelstein and further developed by O. Rossler and associates [65]. In distinction with the present work, this school constructs the inner perspective of the interface which would correspond to the fusion of object and subject, but treating the subject qua object, keeping thus the Cartesian cut. The main difference with the present perspective, is that these authors do not seem to place any regards on self-reference nor the Klein bottle (and thus the transformation between 'inside' and 'outside' cannot find a frame) and the logical processes of the mind, but rather in physical models of this interface starting from the 'inside'. In particular, in the work of Finkelstein in higher order logic, self-reference is untreated [65]. For a critique from an Aristotelian dualist perspective of the work of Saniga the reader is referred to [70]. Of course, the critique that endophysics might elicit -but rather unfairly-, is in terms of solipsism, which is the other side of the face of the positivism ascribed to the designative value imposed to the world by Aristotelian dualism, as we already discussed at the start of this article. Whatever, we cannot overstate the importance of these works towards the constitution of an integrated understanding of the world as a participatory field, very much like the view nurtured by Spinoza.

a cognition operator on cognitive states to a spin operator on two-states spin vectors), relative size, shadow casting, etc., the most important is believed possibly to be stereoscopic vision, i.e. the image formed by the joint use of two eyes. It appears that stereoscopic vision only leads to the formation of three-dimensional images if the two eyes actually sense asymmetric images for each of them, in the contrary there is no distinctive image but a blank homogeneous state [31]. This indicates that the actual concrete perception of a geometry requires an inhomogeneity at its basis; we shall encounter this on discussing torsion geometries). This is most remarkable since stereoscopism is the basis for the conceptual emergence of symmetry with which physics is constructed, and this findings points out that this is only possible for actual asymmetries (which conceptually are based on the manifestation of differences) which if lacking only a void homogeneous perception is formed, i.e. no structured perception of inhomogenities, only the triviality of sameness. This perceptual void where no distinctions are present is the one that is associated with the physical symmetrical vacuum, which as we know from reality it is imaginary, since the vacuum is not exclusively symmetrical but the source of all forms and hence of symmetries in particular. So here we have the appearance that depth to be perceived as an original dimension, a difference that makes a differences is necessary as in the conception of cybernetist G. Bateson, and this is the basic asymmetry between the images of each eye [5].

Philosopher and physicist Patrick Heelan (student of Heisenberg) focused in his work on perception and the geometry of visual space [19] following the geometrical approach to this issues in the pioneering work of Luneburg carried out at the Optics Laboratory at Columbia University in the forties and fifties [46]. Luneburg discovered that hyperbolic Lorentzian metrics may describe the geometry of vision instead of the Euclidean geometry in certain phenomenae, and that a psychometric function dependent on the observer is crucial to visual perception. Furthermore, that time dilation and space contraction are related to the velocity of eye motions and that the baud rate of processing is also a determinant factor in this Lorentz-Fitzgerald phenomenon. ¹¹ The conclusions of these researchers is that there is no

¹¹The reality of this interior geometry of visual representation is testified by the well known fact that pilots flying near to the surface at high velocities are prone to crashing their planes due to a fixation phenomenae which is produced by the high baud transference of information due to the actual Lorentz-Fitzgerald space contraction and time dilation

”pure objective” cognition of an object: Visual perception and cognition depends on contextual interpretations by the subject. Furthermore and most surprisingly, they depend on cultural ¹² and theoretical constructs, to some extent. ¹³ Thus, the classical Cartesian formula is untenable and perception is not secondary to cognition [69] and the subject is a full participant in the construction of his visual model, as in the classical formula for which space and time are pure reflections. This ’interiorization’ of the geometry by the subject with its dual operation of projection for the construction of the ’exterior’ geometry of space and time can be still be linked with the metabolic rate of the production of ATP (adenosine triphosphate) in the brain’s visual area which is still linked with the quantity of light absorbed in the retina [18].

2.2 Light as a fusion of object with subject

Continuing with the previous discussion, if thus time is subjective while at the same time coparticipant with space and in the blending of object with subject, as the philosophical dialectical phenomenology argues, in the concrete realization of this stands the process of photon absorption by the subject. But the photon is no ’external’ particle to the subject, as the classical formula would like us believe but is the physical ’particle’ which is the core of the self-referential process of fusion of object with subject (inasmuch as the geometrical fusion of object with subject for this process is the Cartan torsion as we shall discuss later). Indeed, when we visualize a photon, we are actually visualizing our seeing of the photon, the absorption process by which we complete the objectification of the photon as an independent emitter, object-in-space-before-subject which now when absorbed becomes the fusion of object-with-subject. Placed in a self-referential context, which as we see is the real context, the photon (as a lifeworld) is the observation of

associated to hyperbolic vision.

¹²In the issue of perception, it has been found the existence of universals in cultures’ basic colour vision dynamics [6]. This was the starting point for the development of the so-called prototype theory in cognition theory [67].

¹³Luneburg pioneering findings were developed into geometrical models in Mathematical Psychology - a discipline which was initiated by Weber, Hubel and von Helmholtz- which has confirmed the subjective-objective character of the constructions of the geometrical representations of visual space [33].

the link between the photon (as an element of objective reality) and the perception of the photon (we shall return to this in relation with the Fibonacci type reentrance of a form into itself in the calculus of indications). Thus, we venture the argument that the photon has the same structure that the subject qua object, it is an extended 3-dimensional (with incorporated lower dimension) singularity, at least so with regard to algorithmic semantics [35], inducing compatibility necessary for interaction. Thus, if we follow V. Fock in his critique to General Relativity [14], treating the wave character of photons through the system of equations given by the eikonal equation in a 4-dimensional spacetime provided with a Lorentzian metric g and the wave equation defined by this metric, i.e. the system given by the wave and eikonal equations

$$\Delta_g f = 0, \quad (\text{grad } f)^2 := g^{\alpha\beta} \partial_\alpha f \partial_\beta f = 0 \quad (1)$$

where $\Delta_g f := \frac{1}{|\det(g)|^{\frac{1}{2}}} \partial_\alpha (|\det(g)|^{\frac{1}{2}} g^{\alpha\beta} \partial_\beta f)$ is the action of the Laplace-Beltrami operator on f . We have the following structure of solutions

$$f = \alpha + \rho(G(\Theta)\vec{i} + H(\Theta)\vec{j} + K(\Theta)\vec{k}), \quad (2)$$

where α, ρ, Θ are real valued solutions of the system of eqs. (1) satisfying further

$$g(d\Theta, d\alpha + id\rho) = 0, \quad (3)$$

with G, H, K being real valued functions defined on the manifold satisfying further $G^2 + H^2 + K^2 = 1$; here $\vec{i}, \vec{j}, \vec{k}$ are the standard real quaternions and i is the scalar imaginary square root of -1 . (Unfortunately the proof is too long to be included here [61].) From this it is possible to see that the propagating lightlike particles characterized by the solutions (2) of the system of equations of wave propagation and the eikonal equation (a nilpotence equation [66]), including the photon, are singularities of dimension 3 which contain lower dimension singularities, making the usual figure of the pointlike unextended photon untenable in this perspective [61]. In this context, quantum jumps play an essential role since they represent what is present to cognition and perception, i.e. differences, as we also know well from physics and vision [31]. In terms of this system of equations, we have characterized quantum jumps in terms of singularities of the torsion potential described by the differential of the logarithm of these scalar fields produced by the node

set of them [61].¹⁴ Returning to the characterization of the photon as an 'interface' by which object and subject are united, this interiorization of the 'objective' photon may be in a dual relation with the exteriorization of endogenous light in the subject given by biophoton emission, establishing a flow of in-formation which constitutes the lifeworld of the subject; this endogenous light is presently an issue of great interest in the research of coherent quantum states in biology [8]. This seemingly duality might be linked to a nondual character of interior and exterior as the interface of both, and its topology is, as we shall see is that of the Klein bottle. It is most remarkable that the Klein bottle can be found already as the solution of representation of topographic maps on the neurocortex. As for the interiorization of the visual field in the neurocortex, it is a well known fact that there is a distorted map of the body surface in somatosensory cortex, known as the 'homunculus', and that in the visual cortex there is an orderly map of visual space; furthermore, symmetry properties of simple cell receptive properties lead naturally to the construction of the Klein bottle [79]. So the geometry of visual space has a representation at the visual cortex, and furthermore, at the fundamental level of cells, the topology of the Klein bottle is naturally present. Furthermore, the topographic representations are arranged topologically, and most remarkably, there is experimental evidence that supports that these maps can be represented by the Klein bottle [79,80]. Interesting enough, the starting point is the 2-dimensional Gabor function (of importance in holography [44]) commonly used to model the receptive-field profiles of simple cells [47], which make up a substantial percentage of visual cortical neurons. This function has associated with it an orientation and a phase angle, which, like direction preference, is cyclic over a 360 degrees range. Any combination of orientation and phase angles can be plotted as a position in a rectangle on a two-dimensional plane; if the edges of the rectangle are joined up, then a surface across which these parameters vary continuously is formed. The only way to do this, however, is to construct the Klein bottle. This result allowed to investigate the likely properties of a cortical map in

¹⁴Research on the applications of torsion to a theory of gravitation were initiated in joint work of Einstein and Cartan in the decade of the 30's. In the author's work torsion geometry appears as the natural geometry for unification of spacetime structures, Brownian motion, quantum mechanics and fluid-dynamics [60,62]. This geometry has appeared recently in the work of Horwitz et al [30] on the geometrical structure of Hamiltonian chaos.

which receptive-field orientation and spatial phase vary smoothly. The technique used to do this comes from topology, namely homotopy theory, and considers the behaviour of a loop drawn on the surface of the Klein bottle [80]. So a topological representation linked to the Klein bottle is already present at an holographic representation of vision of the neurocortex. We shall see that these topological representations of the neurocortex, i.e. part of the brain, are also fundamental to the mathematical functionality of the mind through logic, establishing an interesting link between the brain and its inner topological and functional structure, with the mind. ¹⁵

2.3 Higher Order Self-Reference: the Klein Bottle

Returning to the Cartesian cut, while it assumes a continuity of space as a container broken by the object but which paradoxically keeps this continuity, quantum physics and the photon point out to an essential discontinuity of this space which becomes essential when constructing quantum gravity below the scale of Planck's length. In this scale, the very structure of spacetime is at stake jointly with the cognition and perception of the subject on it. So as argued by Rosen [64], it is already in quantum gravity that the Cartesian cut is challenged. Returning to Merleau-Ponty's depth we recall its fundamental property of internalizing the relations among object, subject, and space, alike Gunther's critique to Aristotelian dualism for its elimination of time in failing to provide for it an ontological locus (and further its trivialization), calls for an internalization which would further include time and the laws of thought, which are embodied by logic, taking in consideration more general than Aristotelian-Boolean logic, for which the superposed states of quantum physics are left unconsidered. Indeed, superposed states and entanglement, appears to be the actual world of the quantum domain and this requires logics in which states such as true *and* false, and still, neither true nor false, are considered, as already proposed by Nagarjuna; for a study on the cultural and sociological aspects of time and its relations with logic from a physicist and scholar point of view, we refer the reader to the work of C. K. Raju [59].

The question that we shall address in the following in this article, is the

¹⁵Note added in proof. For recent studies in a setting different to the present one - yet which acknowledges the need of self-reference- on quantum mechanics, cognition and perception, we refer to Conte et al [84] and references therein, notably A. Khrennikov's work.

mathematically formalized presentation of a conception (without neglecting the philosophical aspects) in which object, subject, space and time, quantum physics and multivalued logics have a common ground, which is that of paradox and self-reference, which the latter have not been incorporated in the current studies of quantum physics nor in quantum computation in which a matrix representation of logic is the core mathematical element [4]. We shall present in the course of this work a first approach to this view, in terms of the very paradoxical structure of the laws of thought and the topological structures of in-formation associated to them (which will appear to be non-orientable surfaces such as the Moebius bands and the Klein bottle), and still with regards to the protologic and multivalued logics associated with the calculus of indications by George Spencer-Brown, as well as the matrix logic due to August Stern [76,77], and still its relations with torsion and time. In the presentation of this conception, we shall see that the Klein bottle is the embodiment of a self-referential topology and the working of the mind and reality as a whole. Already in Rosen's work we find that he claims that the origins of his thought on self-reference and paradox can be established in terms of a topological phenomenology that is traced back to Merleau Ponty and Heidegger, which he establishes in terms of the Klein bottle. In contrast with Aristotelian dualism this (genus 0 Riemann) surface is both open and closed, continuous and discontinuous, inside and outside are fused since it has a single side. This surface can be seen -in a first approach- as having an uncontained part (the subject), a contained part (the object) and a containing part (space) [40]. Yet, there is an in-formation flow from the uncontained to the contained part which we could inquire if it can be reduced to the Cartesian cut view of space as a container. In distinction with the Moebius band which is another surface of paradox which can be actually constructed in three-dimensional space by taking a band, twisting it and glueing its opposite extremes and thus we have a surface of paradox which is non-orientable (another important property of the Klein bottle and the multivalued logic that surges from this) but it can still be thought as satisfying the classical figure of object on space independent of the subject.¹⁶ Yet, the Klein bot-

¹⁶Yet we remark that in the case that we rather start with a closed on itself orientable surface, to be able to produce by a twist a Moebius surface we must start by cutting it (i.e. by actually introducing a dislocation, which is geometrically represented by a torsion field [41]), so that in this case the classical figure is produced in neglecting this previous act which does not conform with the figure of object-independent-of-the-subject.

tle which is constructed by identifying two sides of a rectangle by glueing two opposite sides with the same orientation and identifying the other sides having *opposite* orientations which makes this figure impossible to be actually constructed in $3D$ in distinction with the Moebius band and which thus can be seen as contained in space. This produces a topologically imperfect model in $3D$ since a hole has to be produced so that its construction already introduces singularities which then through the in-formation flow produces the whole structure, so that the whole structure is produced from a hole, and this returns to the singularity to complete the flow. ¹⁷ The Cartesian mindset attitude to this would be to view the Klein bottle as embedded in $4d$ where the hole is no longer necessary and in doing this, the concrete real figure is cast into an idealization which cannot be manifested by the subject (the Cartesian minded mathematician ¹⁸) who thus keeps detached from this abstract ideal in-formation now ideally contained in the Cartesian view. ¹⁹ Rosen's stance to which we adhere is instead to keep the hole -so that singularities are unavoidable as in quantum physics or already in the geometrical model of the photon we briefly presented above- as the starting point for questioning the Cartesian stance. In distinction with a Moebius band, a torus or any other object in $3D$, the loss of continuity of the Klein bottle is necessary showing that $3D$ space is unable to contain the surface in the Cartesian stance avows for ordinary objects. So instead of abstracting

¹⁷This is related to D. Bohm's holomovement and the integrality of the paradoxical structure to his implicate order, while the singularity is related to the explicate structure, in a first approximation. Indeed, in this wholeness surging from a singularity and back to it, what is at stake is the integral structure from which the implicate and explicate orders are instances and interchangeable through the flow [7]. In this sense, our repetitious expression of fusion of object and subject should not be a conceived as a mere reparation of the Cartesian cut, but rather an indication of the integral structure of which both are instances in a process in which they are inseparable. Indeed, this is an holographic structure, and as such is constructed by the neurocortex as we already discussed before.

¹⁸Transarchitect and cybernetist Marcos Novak speaks thus of the loss of the "innocence of space", where 'innocence' is a qualification for the Cartesian mindset, as he learnt through the construction of his virtual 'liquid' architecture through algorithms, showing that science and art are unified through *tekné*, in which the subject is integral to this unification [54].

¹⁹We shall see below, the calculus of indications of Spencer-Brown as another setting for an in-formation flow in which object and subject are fused and which produces also time waves as a resultant of a paradoxical equation, that of an uncontained form which contains itself, which we shall argue now, is the case of the Klein bottle.

by incorporating a fourth additional dimension (as is the proposal of Special and General Relativity) we keep the hole that produces the wholeness and instead of an additional dimension we think of the depth dimension as the primal dimension which becomes the source for the Cartesian dimension.²⁰ To resume, the Klein bottle instead of being contained in space it contains itself and the flow of in-formation that is associated to this topology, is the manifestation of this self-containence, this paradoxical situation which becomes real through the production of a singularity which produces the whole structure. By doing this, it supersedes the Cartesian cut and the Aristotelian dualism, by superseding the dicotomy of container and contained, and in semiotic terms, of interpreter and interpreted. We can additionally discern from the previous discussion, that the in-formation process of self-reference, i.e. of consciousness which through the laws of thought which are not longer those of Aristotelian dualism, transforms the 'outside' into the 'inside' world (this transformation is the fourth ontological locus that Gunther proposed: thought as a process), and this transformation has to produce a relation between the laws of thought and the laws of the physical world, and superposition has to be related to this relation and with the actual process of transformation of the 'outside' and 'inside' realms. Since discontinuity can be seen as the source for wholeness, one can enquire on the role of quantization associated to the topology of the Klein-bottle and the in-formation process that is associated to this singularity and the self-referential topology, and still o the role of quantization with regards to the multivalued logic that is associated to this in-formation structure and its paradoxical character. We shall deal with these questions below. Already Rosen established a link between the Klein bottle and quantization and still with Musès hypernumbers which incorporate not only non-trivial square roots of -1 but also of $+1$, the latter being associated to spinors as elaborated by Elie Cartan [9], and more concretely, with the Pauli matrices of quantum mechanics (Rosen, 2008) [63,64]), further applied to a cosmology placed in terms of the hypernumbers which are positive square roots of $+1$.

²⁰We shall see how this is related to the calculus of indications in which depth produces the time and space dimensions as a resultant of paradox, and also, to a gestaltic spin and still a time operator in matrix logic.

3 The Original Distinction, Spencer-Brown's Calculus of Indications

3.1 Torsion as a Self-referential Construction of a Distinction

We assume that we are given a plane surface. In this surface there is no sign, no distinction, just the undifferentiated homogeneous plane, an unbounded world of pure potentiality. As such, this plane corresponds to the world of the subject qua subject and to the lifeworld of all potential phenomena. This undifferentiated state is the plenum, what has become associated with the vacuum state of quantum theory, and in some perspectives, as the void. In the millenary Indian scriptures of the Rg Veda, "...neither exists nor non-exists". We want to proceed to establish a distinction in this plane albeit one with respect to which we can define an outside and an inside. How would nature, in the cognition of a subject, establish such a distinction? This is the question that the scientist in the Cartesian-cut discourse would pose itself. This scientist will surely establish the geometric mode of representation to this basic problem which is nothing else that how can something come out of no differentiated thing. This coming out requires an action, an operation, and since in principle nature may provide this through establishing a distinguished sector, which in one side would establish self-referentially an inside and an outside and the flow through the boundary which the distinction constructs. This plenum we are referring to is about a complete breaking of symmetries, which is masked through the homogeneity of the original state. To ascertain that it is symmetric, would require what we have at our disposition at this stage of our presentation, a way to establish a distinction with respect to which symmetries and asymmetries might be confronted to, so lacking this possibility this plenum contains all possible symmetries and their breaking and still the complete lack of symmetries altogether. So let us take the elementary way of providing a way of (parallelly) translating an infinitesimal vectorfield from one point to the other of a manifold. This geometrical notion of parallel transport essentially preserves the identity of the object translated so in this sense it is prior to a distinction in this context. It

instead requires the possibility of a motion from an original point on which the infinitesimal vector is placed (as a member of the tangent space at the original point) along a curve which is assumed to be smooth or still a continuous yet non-differentiable random path. The possibility of this motion is then linked with the existence of this curve along which the parallel transport is practiced and in this primeval sense the curve and the motion act as the substrate on which the vector is translated. We now take a second infinitesimal vector linearly independent of the first standing at the tangent space of the origin and we translate it along a curve distinct from the first one. We now parallelly translate the original curve and its infinitesimal vector to the end of this second vectorfield and the second vectorfield to the end point of the vectorfield along the parallel transport of the second curve [23]. We now compute the difference between the parallel transported vectorfields; we note en passant that a difference in arithmetic and algebraic terms is the most primitive operation of distinction in this geometrical framework. There are two possibilities, namely: 1) the difference is 0 and then an infinitesimal parallelogram does close creating thus the distinction in terms of which an outside-inside can be established in the plenum given by the unmarked plane. This is the case of a distinction established in terms of a trivial difference, the vanishing of the torsion tensor. Self-referentially, this distinction is trivial, it is constructed in terms of an operation that introduces no difference but the distinction itself. This is the case when the parallel transport is carried out with the unique metric connection with null torsion, which is the Levi-Civita connection defined by a metric on the plane. 2): The case we are pointing to is the one in which the distinction in the plane, i.e. the establishment of an outside and an inside from the completion of a non-closing infinitesimal parallelogram arises precisely from the non-vanishing torsion tensor which allows to complete the would-be yet impossible parallelogram ²¹ with an additional object (functionally, in the Klein bottle the essential singularity, the hole from which the whole is created and the holomovement of which both are only instances) which makes the figure a pentagon and arises from the non-null difference of the final parallelly transported vectorfields, where now the connection to carry out the transport is not the Levi-Civita connection

²¹This impossibility is functionally identical to the impossibility of the creation of the Klein bottle in 3D, it indicates the presence of a logo-physical-geometrical field which is the holographic structure in which the object and the subject are jointly constituted, as we argued before.

derived from a metric (which itself can be trivial and yet the connection have non-null torsion) but from a Cartan connection with torsion (for which the component of this connection derived from the metric can vanish or not, a fact that plays no role in this second case). What matters in these connections, is the skew-symmetric torsion tensor which closes the infinitesimal parallelogram [23]. This creates operationally a geometry in terms of difference whose first manifestation is the torsion, and a distinction which creates an inside/outside, a difference that creates a difference, instead of a lack of difference producing a difference through a fictitious homogeneity (recall our previous discussion on perception and cognition as based on the need of inhomogeneities). In the non-null torsion case, the self-referential character of the construction of the geometry and the distinction is the imprint of the fusion of subject with object, and this can be formulated equivalently in terms of an anholonomic basis which yields a non-zero torsion tensor [23].

This situation is already the case of all manifolds given by Lie groups. They can be provided with a Cartan connection whose torsion is precisely minus the structure coefficients of the Lie algebra provided by the commutator of arbitrary infinitesimal vectorfields. So, if X_1, \dots, X_j denote the basis of a Lie algebra, then the structure coefficients C_{ij}^k , with $C_{ij}^k = -C_{ji}^k$ given by the commutator $[X_i, X_j] = C_{ij}^k X_k$, up to a sign define the torsion tensor of this canonical connection [71]. This example makes our previous arguments evident: Indeed, torsion here produces the infinitesimal symmetries from which the actual Lie group is produced by exponentiation. Thus the geometry is produced by the torsion. We shall see that this structure is very general, and appears already in the laws of thought, and will produce from an orientable surface a superposed quantum-like state in a non-orientable surface which is a Moebius band.

3.2 The Calculus of Indications, Self-reference, Paradox, Depth and the Appearance of Space and Time

Having established a self-referential distinction which we related to torsion as a physical (or better say logo-physical in examining later its connections with logic though its manifestation of a fusion of object and subject is already sufficient to call it likewise) we want to establish the laws of logic which follows from it when we make some very elementary hypothesis on the

distinction mark that the torsion vector has produced in the plane. Having produced self-referentiality the distinction, we shall now abstract its origin related to a Cartan connection (cf. our footnote no. 3 on the issue of semiotic codification argued by Pattee [55]), although we shall encounter it later as the commutator of the True and False logical operators of matrix logic, but for the calculus of indications (again, as argued by Pattee in the context of semiotics [55]), all distinctions are alike though it incorporates at its foundation the idea that the distinction and the subject are exchangeable, the terms of a fusion of object with subject as we shall discuss further below. This is the calculus of indications of G. Spencer-Brown [74] and the multivalued logic which follows from a paradoxical equation in this frame.

Let us return to the role of a distinction which is to establish a very important form of duality: inside/outside, so it is no surprise that the calculus of indications established in terms of it and the mark on the plenum (we prefer this term to 'vacuum') which is an undifferentiated plane, admits a particular interpretation which is none other than Boolean logic but by considering higher-order equations leading to paradox, we shall see that it yields a 4-state logic and later, matrix logic which also has 4-states and still allows for quantum states . This calculus of distinctions cannot exist without its valuation so that the distinction becomes an indication (basically, to be in either side of the mark). Now the basic dynamics is constructed in terms of transversing and not transversing through the mark which indicates a boundary; this creates a cognitive operator which informs the dynamics of the forms with relation to the side of the mark, and is a precursor of the cognitive operator we shall find below for the case of matrix logic with continuous states, and yet, as we shall see, the discrete case of the calculus of distinction is sufficient to generate the cognitive operator in the former case of continuous logical states. This leads to very complex dynamics in which the depth variable plays a central role in establishing the complexity of the forms that can be written in terms of juxtaposition of the marked and unmarked states. Remarkably it was B. Russell (Spencer-Brown's interlocutor at Cambridge Univ.) in spite of being a staunch opposer of dealing with paradox in logic he did reckon that that the actual constituents of logical propositions are forms (this was later elaborated further by Gunther, but with no findings of evident operational applicability, till today [17]); see B. Russell, page 128 [68]). Based on this epistemology, Spencer-Brown reframed this idea with the intention of realigning logic with mathematics and providing a basis for

it, starting by recognizing that Boole had produced his algebra to fit Aristotelian logic, but leaving aside its arithmetic, so he set himself to explore this untrodden territory of studying the primeval non-numerical arithmetic of forms. Spencer-Brown's theme was "that a universe comes into being when a space is severed or taken apart", which we note is very much the case not only of a generic distinction but of torsion as a geometrical structure, to disrupt a non-distinguishable (in terms of structure) space and thus produce a physical geometry by the introduction of this inhomogeneity. As stressed by F. Varela [81], Spencer-Brown's conception "amounts to a subversion of the traditional understanding on the basis of descriptions. It views descriptions as based on a primitive act (rather than a logical value or form), and it views this act as being the most simple yet inevitable one that can be performed. This is a nondualistic attempt to set foundations for mathematics and descriptions in general in the sense that object and subject are interlocked. From this basic intuition, he builds an explicit representation and a calculus for distinctions". As for Spencer-Brown's conception on this fusion, " ... we know see that the first distinction, the mark, and the observer are not only interchangeable, but, in the form, identical". In Spencer-Brown's epistemology, all distinctions in their fundamental sense are alike giving rise to the idea of a primary distinction and an indicational space. All qualitative differences of the criteria and physical or whatever origin (cf. footnote no.3) of the distinction are erased and henceforth reduced to their essential quality: that of generating a boundary in whatever domain. As Varela resumes in his recapitulation of the motives for the calculus of indications in the context of autonomous systems in the general theory of systems and cognitive processes (and thus the operation of the subject in its integrity and the mind), "a criterion of distinction is all that is necessary to establish a phenomenal domain in which the unities are seen to operate. In this regard, a rigorous representation of indications serve a double purpose. On the one hand, it gives a foundation for systemic descriptions; I would say that it can be regarded as the foundations of system theory (²² . On the other hand, to start from the foundation of indication is faithful to the epistemology that pervades this presentation, in which the observer-community is always a participant". So having reduced all qualitative differences between distinctions,

²²And it was subsequently applied to social systems and organizations in general, in the work of Luhmann [45] and by the so called sociocybernetists.

the calculus of indications is presented in a minimalistic way:

A distinction is drawn (in physics we associated this to torsion, and further below we shall extend this to the laws of thought). The parts of the space shaped by the distinction are called the states of the distinction. The space and states are called the form of the distinction. The state distinguished by the distinction is marked with a sign, which in the Spencer-Brown original presentation is a sign alike \neg but with an equally lengthed vertical bar which by an uncanny coincidence that is familiar to semioticians [12] represents the fact that in the second state called the empty state precisely because has no symbol associated to it, \square , or also the unmarked state, an impossible parallelogram has been closed (which is the meaning of torsion). Then \neg is called a cross \neg , the concave side of the mark is the inside and further, the mark is interpreted as an instruction to cross the boundary of the primary distinction. Thus the mark is already a generator not only of a flow through it, but also a generation of depth as we iterate the crossing through subsequent marks and more generally of expressions which are given by arrangements of marked and unmarked states, so that the forms \neg , \square , are expressions. Due to typographical limitations we shall denote instead the marked state by $()$. Now the calculus of indications is based on two axioms, one called a form of condensation: $(()) = ()$ and a form of cancellation: $()() = ()$. In the Boolean interpretation of the calculus of indications, the marked state stands for the negation operator, so that the form of condensation is an axiom of nilpotence since the empty state is associated with the false state 0 and the marked state $()$ with 1. With these provisions the calculus that follows is the already mentioned calculus of indications. In this calculus we consider arrangements which can be of great complexity, say for a simple example, the arrangement $(((()))) () ()$ which reduces to $((())) () ()$ and further to $(()) ()$ and finally to $()$. Two methods of evaluation are remarkable and both are related to the depth of the expression given by the crossings. One of them is the previous consisting in looking for the deepest spaces of the expression where there are marks that do not contain other marks. At such places condensation or cancellation may be applied to simplify the given expressions. In the second method one regards the deepest spaces as sending signals of value up through the expression to be combined with a global valuation. To wit, let m stand for the marked state and n for the unmarked state. Thus, $mm = m, mn = nm = m, nn = n$ and $(m) = n, (n) = m$. We now use these

labels as signals in the following example:

$$\left(\left(\left(\left(\right)\right)\right)\right) = \left(\left(m\left(\right)\right)\right) = \left(\left(n\left(m\left(\right)\right)\right)\right) = \left(m\left(n\left(m\left(n\right)\right)\right)\right) = n = . \quad (4)$$

This procedure starts from the deepest spaces and labels those values that are unambiguous until a value for the whole expression appears, in this case the unmarked state. Both methods are compatible, and they both reflect the nature of the cross being both a (self-referential) operator and an operand, which is another expression of the blending of subject with object, respectively. As an operator, the expression filters its own inner signals, creating a pattern or waveform that culminates in its evaluation, i.e. its reduction to its most simple form which is always existent. As Varela remarks "at this level, between a form (or figure) and its dynamic unfolding (or vibration) can be seen. This reflects the complementarity invariance-change, or space-time, which is to be encountered at many points in science. For our consideration of circularity (self-reference) keeping this complementarity in mind is the key to resolving what have usually appeared to be vexing paradoxes." (page 113 [81]). It is clear from the topological perspective that the primeval dimension being depth is already transparent in the calculus of indications, and it becomes the source for the space and time dimensions, as we shall see below.

So far we have only considered the relation of containment, that is we have only considered the inside/outside relationship between crosses. If we viewed this in $3D$ we would have a chain of Chinese boxes with very complex arrangements. In systems theory when considering autonomous systems their organization contains bootstrapping processes that exhibit indefinite form recursion of their component elements. This would be a form that reenters its indicational space, and thus in-forms itself. The geometrical-topological example we already encountered is the Klein bottle. A way for expressing this reentering is to say that a form, say, f , is identical with parts of its contents: $f = \phi(f)$ where ϕ is some indicational expression containing f as a variable, which is a self-referential expression. For example $f = ((f))$ which follows from the axiom of cancellation. A very important example by Kauffman is the expression $f = ((f)f)$ which leads to the Fibonacci sequence [37]. This example points already to the fusion of object and subject. Indeed, if we denote by $[X]$ observing X , and $X[X]$ as the link between X and $[X]$ then $[X[X]]$ is the observation of the link between X and observing X , then we can say that I (alternatively, the photon, both as a fusion of subject with object)

am the observation of the relation between myself (as a subject qua subject) and observing myself (self-referential view of the subject qua object).

Let us examine the following example of this reentrance:

$$f = (f). \quad (5)$$

which in terms of the containment relation is the uncontained form which is identical to its containment, the Klein bottle. If we place $f = \text{ } ,$ the unmarked state, then we get in replacing in the paradoxical eq. (5) that $\text{ } = \text{ } ,$ the unmarked state is equal to the marked state, an absurd. If instead we take $f = \text{ } ,$ the marked state, we now get $\text{ } = \text{ } = \text{ } ,$ absurd as well . We can examine this paradoxical equation in two different ways: it is either a pattern or a form, or it is a dynamics in time, and here the or is not exclusive, and points to the fact that the depth variable is implied already by the r.h.s. of the paradoxical equation already in the first step of the cross, is the primeval dimension for both space and time. If we look at this as a pattern, or a value, not reducible to marked or unmarked, which we can denote by the sign of the Ouroboros (unfortunately, we cannot here provide it), the serpent that bites it tail -this is a crucial example of Pattee's conception of the sign already discussed-, or we can view it as a prescription for recursive action: $f \rightarrow (f),$ and thus we obtain the sequence

$$\text{ } \rightarrow \text{ } \rightarrow \text{ } \rightarrow \text{ } \rightarrow \dots, \quad (6)$$

which can be seen as regeneration constantly an oscillation that is identical to parts of itself:

$$f = \text{ } = (f), \quad (7)$$

which is the spatial interpretation and is the Klein bottle (but for which the deepest space in f has become indeterminate and is not evident how to carry out the calculations), and which due to the form of cancellation the oscillation

$$\mathcal{I} := \text{ } \rightarrow \text{ } \rightarrow \text{ } \rightarrow \text{ } \rightarrow \dots, \quad (8)$$

or instead the different half-period displaced oscillation so that the 'first' state below is the unmarked state:

$$\mathcal{J} := \text{ } \rightarrow \text{ } \rightarrow \text{ } \rightarrow \dots \quad (9)$$

corresponding to the associated recursive dynamics which unfolds to two temporal oscillations between the marked and unmarked states. It is important to stress that the double nature of self-reference, blending operand and operator, cannot be conceived outside of time as a process in which two states alternate. This leads to the peculiar equivalence of self-reference and time, as self-reference cannot be conceived outside time and time comes in whenever self-reference is allowed. As further stressed by Varela one can consider reentry as one kind of periodicity of descriptions of any domain, and in the particular case of the Klein bottle, reentry is the way that the whole becomes the hole that becomes the whole, and so on. If we restrict ourselves to the Boolean interpretation then

$$\mathcal{I} = \dots 1 \rightarrow 0 \rightarrow 1 \rightarrow 0 \dots, \quad (10)$$

$$\mathcal{J} = \dots 0 \rightarrow 1 \rightarrow 0 \rightarrow 1 \dots \quad (11)$$

If we have an arbitrary pattern $[a, b] = \dots ababab \dots$ and define $([a, b]) = [(b), (a)] = \dots (b)(a)(b)(a) \dots$ then we have that $\mathcal{I} = (\mathcal{I})$ and $\mathcal{J} = (\mathcal{J})$ are both solutions of the paradoxical eq. (5), so that these time oscillations under this interpretation of the action of a distinction on them as defined by an ordinary inversion and a half-period shift. Note also that $\mathcal{I}\mathcal{J} = ()$ since at any step either \mathcal{I} or \mathcal{J} are marked.

Returning to the paradoxical equation $f = (f)$ if we use the Boolean interpretation in which $()$ becomes the negation and we think of f as a Boolean variable x , then (x) is $-x$ and the equation becomes that both x and $-x$ are true, i.e. the product $x(-x)$ equals 1, the true state of Boolean algebra, or still $x^2 = -1$, which has for solution any square root of minus [26]. It was Spencer-Brown's genius to extend logic to include imaginary numbers, which in the context of our discussions are related to time and appear as the oscillations I and J . Thus, the Klein bottle is naturally related to hyper-complex numbers, yet, as we shall further see, it is also related to non-trivial square roots of $+1$, yet in a different way that the one proposed by Rosen [63](2008). We shall now pursue another line of research. Consider the space $\hat{B} = B \times B = \{[a, b] | a, b \in B\}$ with the operations

$$[a, b] + [c, d] = [a + c, b + d], [a, b][c, d] = [ac, bd] \quad (12)$$

Furthermore we identify $a \in B$ with $[a, a] \in \hat{B}$ If B is an algebra of two states $()$, with the properties of the forms of cancellation and condensation,

and $([a, b]) := [(b), (a)]$, then \hat{B} is a complete algebra [66]. If we further define the states $\mathbf{0} = [\quad, \quad]$ (the waveform of unmarked states) and $\mathbf{1} = [(\quad), (\quad)]$, the waveform of marked states, which in the Boolean interpretation of the calculus of indications amounts to $\mathbf{0} = [0, 0]$ and $\mathbf{1} = [1, 1]$, where 0, 1 correspond to the false and true states respectively, then together with the states $\mathcal{I} = [(\quad), \quad], \mathcal{J} = [\quad, (\quad)]$ which in the Boolean representation amount to

$$\mathcal{I} = [1, 0], \quad \text{and} \quad \mathcal{J} = [0, 1], \quad (13)$$

then \hat{B} becomes a four-valued de Morgan algebra [36]. Furthermore, $\mathcal{I}\mathcal{J} = \mathbf{0}, \mathcal{I} + \mathcal{J} = \mathbf{1}$.

We wish to show now how in this setting we can build a model of space-time and of Lorentz transformations, following [38] but with an important difference to the usual construction in physics, since a non-trivial square root of +1, an hypernumber associated to spin, will appear acting on the space coordinate x , of a two-dimensional spacetime. We note that in particular, if a, b would stand for numbers, say $a = t - x, b = t + x$, then $[t - x, t + x] = [t, t] + [-x, x] = t[1, 1] + x[-1, 1] = t\mathbf{1} + x\tilde{\sigma}$, where $\tilde{\sigma} = [-1, 1]$ is a non-trivial square root of $\mathbf{1}$; indeed $(\tilde{\sigma})^2 = \mathbf{1}$. We shall later see that $\tilde{\sigma}$ corresponds to a spin operator, and in fact to an hypernumber which is a Pauli operator [49]. In this case $[a, b] = [t - x, t + x]$ correspond to the radar coordinates of a two-dimensional spacetime whose points have the group property $[a, b][a', b'] = (t + x\tilde{\sigma})(t' + x'\tilde{\sigma}) = (tt' + xx') + (tx' + t'x)\tilde{\sigma}$, similar to complex numbers but instead $(\tilde{\sigma})^2 = \mathbf{1}$.

If we consider further a waveform of the form $[k, k^{-1}]$ with $kk^{-1} = 1$ where $[k, k^{-1}] = [(1+v)/\sqrt{(1-v^2)}, (1-v)/\sqrt{(1-v^2)}]$, then v is the relative velocity of the reference frames and we obtain the usual Lorentz transformations in the form [36]

$$\begin{aligned} [k, k^{-1}][a, b] &= [k, k^{-1}][t - x, t + x] = [k, k^{-1}](t + x\tilde{\sigma}) \\ &= (1/\sqrt{(1-v^2)} - v\tilde{\sigma}/\sqrt{(1-v^2)})(t + x\tilde{\sigma}) \\ &= (t - xv)/\sqrt{(1-v^2)} + (x - vt)\tilde{\sigma}/\sqrt{(1-v^2)} \\ &= t' + x'\tilde{\sigma}. \end{aligned} \quad (14)$$

Consider patterns of the form [38]

$$\begin{aligned}
&\dots abababab\dots \\
&\dots cdcdcdcd\dots \\
&\dots abababab\dots \\
&\dots cdcdcdcd\dots
\end{aligned} \tag{15}$$

Note that the operations of above are natural with respect to this juxtaposition. We know compare this pattern with the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Each matrix freezes out a view to the infinite pattern. To denote this patterns we write

$$[a, d] + [b, c]\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{16}$$

where $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is another non-trivial square root of the identity; indeed $\eta^2 = 1$, where 1 is the hypernumber represented by the 2×2 identity matrix. The effect of η is to rotate an iterant by ninety degrees in the formal plane. We shall later see that η corresponds to the negation operator *NOT* in matrix logic (which plays an essential role in quantum computation as well as its square root [4]), or still, with the Pauli matrix σ_x . We introduce now the delay shift operator which we have implicitly introduced before, defined by $D[x, y] = [y, x] := \overline{[x, y]}$ so that the four matrices seen in the two-dimensional pattern are, in writing $A = [a, d], B = [b, c]$, are $A + B\eta, B + A\eta, \bar{B} + \bar{A}\eta, \bar{A} + \bar{B}\eta$. Finally we can write matrix multiplication in taking in account that $\eta^2 = 1$ and that for any iterant $[x, y], \eta[x, y] = \overline{[x, y]}\eta$. For instance, for $\tilde{\sigma} = [-1, 1]$, define

$$i := \tilde{\sigma}\eta = [-1, 1]\eta \tag{17}$$

so that

$$ii = \tilde{\sigma}\eta\tilde{\sigma}\eta = \tilde{\sigma}\bar{\tilde{\sigma}}\eta\eta = \tilde{\sigma}(-\tilde{\sigma})1 = -1. \tag{18}$$

So we have constructed the commutative square root of -1 in its complex representation

$$i = \tilde{\sigma}\eta = [-1, 1]\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

which is the hypernumber i_1 [49].

We are especially interested for generating matrix logic in the following matrices formed from patterns

$$\mathcal{I} = [1, 0], \mathcal{I}\eta = [1, 0]\eta, \mathcal{J} = [0, 1], \mathcal{J}\eta = [0, 1]\eta, \quad (20)$$

which have the matrix representations

$$\mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{I}\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathcal{J} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathcal{J}\eta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (21)$$

We shall see below that we can generate matrix logic with these four patterns. This is most remarkable since matrix logic is a more richer generalization of Boolean logic which deals only with scalars, and works instead with the 2×2 matrix representation of logical connectives, which then assume the role of operators which may have inverses; furthermore, it incorporates continuous logical variables and allows for connection with quantum mechanics through a bra-ket formalism, incorporating also quantum superpositions. What will turn out most remarkably, is that we shall be able to generate matrix logic from the two imaginary solutions of the paradoxical equation $f = (f)$, namely \mathcal{I} and \mathcal{J} , and furthermore $\mathcal{I}\eta$ and $\mathcal{J}\eta$, where as we said before, η is the matrix form of the negation operator, or still the Pauli matrix σ_x .

4 Self-reference, The Klein Bottle, Torsion And The Laws Of Thought

We have already discussed on the relation between self-reference and torsion, and still that torsion already appears naturally in Lie algebras as (minus) the structure coefficients that appear in the commutation of their elements, or still, the Lie bracket. In this section we shall see that torsion also appears in logic, particularly in a more general form of logic which includes quantum and fuzzy logic as particular cases and is called matrix logic and was developed by August Stern [76,77]. By promoting the "truth tables" of usual Boolean logic to matrix representations, Stern was able to produce a covariant operator logic theory [76] in which logical operators may admit inverses, and the operations of commutation and anticommutation are natural. Furthermore, logical operators can interact by multiplication or addition

and, in some cases, being invertible, they yield thus a more complex representation of the laws of thought than the one provided by the usual Boolean theory of logical connectives. It furthermore can be related to quantum mechanics for two-state systems as we shall describe below, and has fuzzy and quantum logics as particular cases of it [77]. Matrix logic is naturally quantized, since its eigenvalues take discrete values which are $\pm 1, 0, 2, \pm\phi$, with ϕ the Golden number [76]. The picture that stems is that matrix logic can be seen as the self-referential logical code which stands at the foundation of quantum physics to which is indissolubly related. We shall not enter into the subject completely, but will restrict ourselves to the relation between matrix logic, self-reference, non-orientability and the Klein bottle, nilpotent hypernumber representations of quantum fields that represent some logical operators, and the transformation between a cognitive operator that stems from the quantum-classical structure of this logic, in particular when seen from the perspective of the True and False operators which extend the scalar true and false of Boolean logic and have them as eigenvalues, and the torsion generated by them. It is in fact very remarkable that the structures that we have encountered until now would produce the unification that we have just shortly digressed. So, now to work.

If we consider a space of all possible cognitive states (which in this context replace the logical variables) represented in this plenum as the set of all Dirac bras $\langle q| = (\bar{q} \ q)$, where $\bar{q} + q = 1$ ²³, is a continuous cognitive logical value

²³Notice that a difference with the definition of qubits in quantum computation, is that for them we have the normalization condition for complex numbers of quantum mechanics. In this case, the values of q are arbitrary real numbers, which leads to the concept of non-convex probabilities. While this may sound absurd in the usual frequentist interpretation, when observing probabilities in non-orientable surfaces, say, Moebius surfaces, then if we start by associating to both sides of an orientable surface -from which we construct the Moebius surface by the usual procedure of twisting and gluing with both sides identified- the notion of say Schroedinger's cat being dead or alive in each side, then for each surface the probability of being in either state equals to 1 and on passing to the non-orientable case, the sum of these probabilities is 2. While this is meaningless in an orientable topology, in the non-orientable case which actually exist in the macroscopic world, this value is a consequence of the topology. In this case, superposed state 'being alive and being dead' or 'true plus false' which is excluded in Aristotelian dualism, is here the case very naturally supported by the fact that we have a non-trivial topology and non-orientability. (By the way, Musès claimed that logic should be related to topology [59], since intuitively logical connectives establish topological links; though he was not able to frame it mathematically, his intuitions were correct on bringing on this issue and on bringing to the

not restricted to the false and true scalar values, represented by the numbers 0 and 1 respectively. In fact, q can take arbitrary values as we shall elaborate further below. Still, the standard logical connectives admit a 2×2 matrix representation of their 'truth tables' and now we have that for such an operator, L , we have the action of L on a ket $|q \rangle = \begin{pmatrix} \bar{q} \\ q \end{pmatrix}$ is denoted by $L|q \rangle$ alike the formalism in quantum mechanics, and still we have a scalar truth value given by $\langle p|L|q \rangle$, where $\langle p|$ denotes another logical vector. We can further extend the usual logical calculus by considering the Truth and False operators, defined by the eigenvalue equations $\text{TRUE}|q \rangle = |1 \rangle$ and $\text{FALSE}|q \rangle = |0 \rangle$, where $|1 \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|0 \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the true and false vectors. It is easy to verify that the eigenvalues of these operators are the scalar truth values of Boolean logic. We can represent them by the matrices

$$\text{TRUE} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \text{FALSE} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (22)$$

We note that the spaces of bras and kets do not satisfy the additivity property of vector spaces -while keeping the property that one is the dual of the other- due to the fact that normalization is not preserved under addition. A superposition principle is necessary. If $|p \rangle$ and $|r \rangle$ are two normalized states, then the superposition defined as follows

$$|q \rangle = c|p \rangle + \bar{c}|r \rangle, \quad \text{where } \bar{c} + c = 1, \quad (23)$$

also defines a normalized logical state. We can interpret these coefficients as components of a logical state $|c \rangle$ or still a probability vector, termed *denktor*, a German-English hybrid for a thinking vector. The normalization condition is found as follows: Multiply the states $|p \rangle$ and $|r \rangle$ by \bar{c} and c , respectively. By definition, the normalization condition on the sum $|q \rangle$ with coefficients \bar{c}, c leads to

$$\begin{pmatrix} \bar{q} \\ q \end{pmatrix} = c \begin{pmatrix} \bar{p} \\ p \end{pmatrix} + \bar{c} \begin{pmatrix} \bar{r} \\ r \end{pmatrix} = \begin{pmatrix} c\bar{p} + \bar{c}\bar{r} \\ cp + \bar{c}r \end{pmatrix}, \quad (24)$$

fore his hypernumbers; he also claimed that paradoxes appeared as a result of incomplete contextual information and thus were the result of wrong interpretations [50].) As for the case of negative probabilities, we see in the previous example that -1 is the probability value complement of the value 2.

yet, since $\bar{q} + q = c\bar{p} + \bar{c}\bar{r} + cp + \bar{c}r = c(\bar{p} + p) + \bar{c}(\bar{r} + r) = c.1 + \bar{c}.1$ and thus $c + \bar{c} = 1$ since $|q\rangle$ is a normalized state by assumption. So through this superposition principle is that we can give a vector space structure to normalized cognitive states. We now can identify under these prescriptions, the tangent space to the space of bras (alternatively, kets) with the space itself. ²⁴

4.1 Holography, the Liar Paradox and Improbabilities

Logical superposition is recursively closed, in contrast with quantum superposition, and thus self-referential in the sense that two normalized logical states add to a new superposed normalized state and we can continue this indefinitely. This introduces an holographic behaviour in the space of all normalized cognitive states, since any such state can be expanded into a decomposition of all states and all states can be brought to bear on a given one, just by applying the proper combinations under the rule of the superposition principle. This is in sharp contrast with the quadratic rule of normalization of quantum superposition, say, $|c_1|^2 + |c_2|^2 = 1$ while still challenges the positivity assumption on classical probabilities, and still negative probabilities can occur. One way of understanding how these negative probabilities can be seen in introducing the inverse of the logical operators which shows that it is not a matter of a mere sign duality what is at stake for introducing them [76]. An important application of the logical superposition principle is to provide a solution of the Liar paradox. Take a superposition of undecided states $|\frac{1}{2}\rangle$ (probability of true = probability of false) with the initial condition being Boolean states, $|x\rangle$ either $|0\rangle$ or $|1\rangle$, true and false states respectively. Then, we can combine them with a denktor $(2 \quad -1)$ to obtain $2|\frac{1}{2}\rangle + (-1)|x\rangle$ to obtain the state $|\bar{x}\rangle$, the complement state of $|x\rangle$ defined by $|1-x\rangle = \begin{pmatrix} x \\ \bar{x} \end{pmatrix}$. But in the other hand $|\bar{x}\rangle = \text{NOT}|x\rangle$, which is a perfectly meaningful state given by the negation operator NOT acting on a Boolean state, which was thus produced from superposed states with a negative probability. Thus, what in the calculus of indications below will be achieved by taking the imaginary elements \mathcal{I} and \mathcal{J}

²⁴Here it is simple to see that if $|q\rangle, |q'\rangle$ are two superpositions, then for any operator L , $L(|q + q'\rangle) = L|q\rangle + L|q'\rangle$.

solutions of the paradoxical equation identical to the Liar, in matrix logic is achieved by taking denktors with improbabilities. The 'imaginary' element (the Klein bottle its manifestation) has been transmuted into the 'improbable' one, which is uncannily reflected in language, both mathematical and colloquial (this is another instance of semiotic codification of a deeper field, as in Pattee [55]). The new element that classical probability theory does not include, self-reference is here relevant. In classical probability theory there is a Cartesian cut: a dynamical process and the estimation of it in which the subject carrying the estimation is detached on the process and cannot influence it, while in the case of the thinking mind, it is altogether by self-estimation, self-measurement, the process and the estimation are unified. Indeed the work of self-estimation, and if necessary the introduction of an improbability, is done by the denkter.

4.2 Torsion in Cognitive Space, the Klein Bottle, Quantum Mechanics and Field Theory, Logic and Hypernumbers, and the Cognition, Time and Spin Operators

Returning to the vector space structure provided by the superposition principle, and thus the identification of its tangent space with the vector space itself, it follows that a vector field as a section of the tangent space can be seen as transforming a bra (ket) vector into a bra (ket) vector through a 2×2 matrix, so we can identify the tangent space with the space of logical operators. We have as usual the commutator of any such matrices $[A, B] = AB - BA$ and the anticommutator $\{A, B\} = AB + BA$. In particular we take the case of $A = \text{FALSE}$, $B = \text{TRUE}$ and we compute to obtain

$$[\text{FALSE}, \text{TRUE}] = \text{FALSE} - \text{TRUE}, \quad (25)$$

$$\{\text{FALSE}, \text{TRUE}\} = \text{FALSE} + \text{TRUE}. \quad (26)$$

Thus in the subspace spanned by TRUE and FALSE we find that the commutator that here coincides with the Lie-bracket of vectorfields defines a torsion vector given by the vector $(1 \ -1)$, and that this subspace is integrable in the sense of Frobenius: Indeed $[\text{FALSE}, \text{TRUE}], \text{TRUE}] = [\text{FALSE}, \text{TRUE}]$ and $[[\text{FALSE}, \text{TRUE}], \text{FALSE}] = [\text{TRUE}, \text{FALSE}]$. Furthermore, on account that

$\text{TRUE}^2 = \text{TRUE}$ and $\text{FALSE}^2 = \text{FALSE}$, i.e. both operators are idempotent, then the anticommutators also leaves this subspace invariant.

The remarkable aspect here is that the quantum distinction produced by the commutator, exactly coincides with the classical distinction produced by the difference (eq. (25)), while the same is valid for the anticommutator with a classical distinction which is represented by addition (eq. (26)). We notice that in distinction of quantum observables, these logical operators are not hermitean and furthermore they are noninvertible. Furthermore, we shall see below how this torsion is linked with the creation of cognitive superposed states, very much like the coherent superposed states that appear in quantum mechanics. Now, if we denote by M the commutator $[\text{FALSE}, \text{TRUE}]$ so that from eqs. (22, 25) we get

$$M = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad (27)$$

we note that it is nilpotent, (in fact a nilpotent hypernumber, since $M = \epsilon_2 + i_1 = \sigma_z + i_1$)

$$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \equiv \mathbf{0}, \quad (28)$$

thus yielding the identically zero matrix, representing the universe of all possible cognitive states created by a non-null divisor of zero, which thus creates a polarization of this plenum precisely through the fact that the torsion is a superposed state which cannot be fit into the scheme of Boolean logic but can be obtained independently by the loss of orientability of a surface which thus allows for paradox. Since M coincides with the classical difference between TRUE and FALSE, which are not hermitean, then we can think of this non-invertible operator as a cognitive operator related to the variation of truth value of the cognitive state, as we shall prove further below that $M = -\frac{d}{dq}$.

We would like to note that this polarization of the plenum $\mathbf{0}$ is not unique, there are many divisors of $\mathbf{0}$, the plenum, for instance the operator

$$\text{ON} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} := a^\dagger, \quad (29)$$

and

$$\text{OFF} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} := a \quad (30)$$

satisfy

$$a^2 = \mathbf{0}, (a^\dagger)^2 = \mathbf{0}, \quad (31)$$

and furthermore, $\{a, a^\dagger\} = I$, so they can be considered to be matrix representations of creation and annihilation operators, a^\dagger and a as in quantum field theory. In fact, if we consider the wave operators given by the exponentials of a, a^\dagger we have

$$e^a = I + a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \text{IMPLY}, e^{a^\dagger} = I + a^\dagger = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \text{IF}, \quad (32)$$

where $\text{IMPLY} = \rightarrow$ is the implication, and $\text{IF} = \leftarrow$ is the converse implication: $x \leftarrow y = \bar{x} \rightarrow \bar{y}$. Thus the implication and the converse implication logical operators are both wave-like logical operators given by the exponentials of divisors of $\mathbf{0}$, and in fact they are derived from quantum field operators of creation and annihilation in second-quantization theory, a^\dagger and a , respectively, which in fact can be represented by nilpotent hypernumbers. Indeed, $a = \frac{1}{2}(\epsilon_3 - i_1) = \frac{1}{2}(\sigma_x - i_1)$ and $a^\dagger = \frac{1}{2}(\epsilon_3 + i_1) = \frac{1}{2}(\sigma_x + i_1)$ [49].

Now we wish to prove that the interpretation of M as the logical momentum operator is natural since $M = -\frac{d}{dq}$. Indeed,

$$-\frac{d}{dq}|q\rangle = -\frac{d}{dq} \begin{pmatrix} 1 - q \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \bar{q} \\ q \end{pmatrix} = M|q\rangle \quad (33)$$

so that for any normalized cognitive state $|q\rangle$ we have the identity

$$M = -\frac{d}{dq}, \quad (34)$$

which allows to interpretate the cognitive operator as a kind of logical momentum. Thus, in this setting which is more general but less primitive than the calculus of distinctions, it is the non-duality of TRUE and FALSE what produces cognition as variation of the continuous cognitive state. We certainly are here with a situation that is far from the one contemplated by Aristotle with his conception of a trivial duality of (scalar) true and false, and which lead the elimination (and consequent trivialization) of time and of subjectivity, as argued by Gunther, which we discussed in the initial phase

of this article. ²⁵

Now consider a surface given by a closed oriented band projecting on the xy plane. Thus to each side of the surface we can associate its normal unit vectors, (1 0) and (0 1). Suppose that we now cut this surface and introduce a twist on the band and we glue it to get thus a Moebius surface. Now the surface has lost its orientability and we can identify one side with the other so that we can generate the superpositions

$$\langle 0|_+ \langle 1| = \langle (1 \ 1)| = \langle S_+|, \quad \langle 0|_- \langle 1| = \langle (1 \ -1)| = \langle S_-|. \quad (35)$$

which we note that the latter corresponds to the torsion produced by the commutator of TRUE and FALSE operators. These states are related by a change of phase by rotation of 90 degrees. What the twisting and loss of orientability produced, can be equivalently produced by the fact that TRUE and FALSE are no longer as in Boolean logic and the Aristotelian frame, they are no longer dual and what matters is their difference, which in the case of scalar truth values does not exist. The other state also can be interpreted as a state that represents the fact that the states as represented by vectors, have components standing for truth and falsity values which are independent, so that the Aristotelian link that makes one the trivial reflexive value of the other one is no longer present: they each have a value of their own. In that case then (0 0) is another state, 'false and true' (which is the case of the Liar paradox as well as Schroedinger's cat), which together with (1 1), 'nor false nor true' state together with (0 1), true, and (1 0) false states we have a 4-state logic in which the logical connectives have been promoted to operators. We have thus returned to Nagarjuna [51]. ²⁶

²⁵This is the upgrade of the notion of value we mentioned before in footnote 5 that comes in several steps. First the upgrading of the logical variables from scalars to vectors, and logical connectives to logical operators where in particular the TRUE and FALSE operators are not in duality and their difference (either classical or quantum) generate a cognition operator; furthermore, the relation between this operator and superposition quantum states as we shall see further below.

²⁶So in the line proposed by Gunther, that the loss of dualism has to do with the introduction of time and subjectivity, we shall examine this aspect together with the loss of the two-valuedness of Boolean logic which is the algebraic expression of Aristotelian dualism. It is clear from the fact that the commutator of TRUE and FALSE is self-referential since it implies a twist of the true and false vectors and the involutiveness of these operators, that subjectivity in its most primitive form, that of self-referentiality, is

Now consider for an arbitrary normalized cognitive state q the expression

$$\begin{aligned}
[q, M]|q \rangle &= [q, -\frac{d}{dq}]|q \rangle = -q\frac{d}{dq}|q \rangle + \frac{d}{dq}q|q \rangle = -q\frac{d}{dq} \begin{pmatrix} 1-q \\ q \end{pmatrix} \\
&+ \frac{d}{dq} \begin{pmatrix} q-q^2 \\ q^2 \end{pmatrix} = \begin{pmatrix} q \\ -q \end{pmatrix} + \begin{pmatrix} 1-2q \\ 2q \end{pmatrix} = |q \rangle, \quad (36)
\end{aligned}$$

for any normalized cognitive state q so that we have the quantization rule

$$[q, M] = I, \quad (37)$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity operator. Instead of the commutation relations of quantum mechanics $[q, p] = i\hbar$ for $p = -i\frac{\partial}{\partial q}$ and those of diffusion processes associated to the Schroedinger equation, $[q, p] = \sigma$ where $p = \sigma\frac{\partial}{\partial x}$ with σ the diffusion tensor given by the square-root of the metric g on the manifold with coordinates x on which the diffusion takes place so that $\sigma \times \sigma^\dagger = g$ [62], we have that the commutation of a normalized cognitive state with the cognitive (momentum) operator is always the identity yielding thus a fixed point. Indeed, consider the function $F_M(q) = [q, M]$, then $F_M(F_M(F_M(F_M(\dots))))(|q \rangle) = |q \rangle$, for any normalized cognitive state $|q \rangle$. Thus, $F_M(q)$ defines what is called in system's theory an eigenform, albeit one which does not require infinite recursion but achieves a fix point already in the first step of the process, by the formation of the commutator $[q, M]$ [39]. This is the structure of the Self, which whatever operation may suffer by the action of logical operators, it retains its invariance by the quantization of logic as expressed by eq. (37).

Now we want to return to the superposed states, S_+ and S_- , the latter being the torsion produced by the commutator of the TRUE and FALSE operators, to see how they actually construct the cognitive operator. First a slight detour to introduce the usual tensor products of two cognitive states,

present. This was the core of the elaboration of psychoanalysis by Jacques Lacan, who also proposed to consider a higher form of self-reference, the one produced by the gluing of two oppositely twisted Moebius surfaces, to yield the Klein bottle, a higher degree surface of paradox [52]. Another relevant and comprehensive recent work in the Klein bottle is due to M.C. Purcell (whose collaborator, amateur scientist S. Morgan accomplished the unique feat of the uni-dimensional diagonal weaving of the Klein bottle), whose work focused in the development of cultural studies related to this surface of paradox and cosmology [58].

$|p\rangle\langle q|$ which as the tensor product of a vector space and its dual is isomorphic to the space of linear transformations between them, we can think as an operator L acting by left multiplication on kets and by right multiplication on bras. So that if $L = |p\rangle\langle q|$ then $\langle y|L|x\rangle = \langle y,p\rangle\langle q|x\rangle$, for any $\langle y| = \bar{y}\langle 0| + y\langle 1|$ and $|x\rangle = \bar{x}|0\rangle + x|1\rangle$, where $\langle x|y\rangle = \delta_{xy}$ equal to 1 for $x = y$, and equal to 0 for $x \neq y$ and $\sum_i |x_i\rangle\langle x_i| = I$. Then,

$$M = |S_+\rangle\langle S_-|, \quad (38)$$

which shows that the cognitive operator that arises from the quantum-classical difference between the True and False operators can be expressed in terms of the tensor products of the superposition states, being the sum of the true and false states and the torsion produced in the quantum commutator of the TRUE and FALSE operators.

Let us now introduce the operator defined by

$$\Delta = a - a^\dagger \quad (39)$$

so that it follows that its matrix representation is

$$\Delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (40)$$

and furthermore

$$\Delta = \rightarrow - \leftarrow. \quad (41)$$

We shall call Δ the TIME operator.²⁷ We notice that it is unitary and antisymmetric:

$$\text{TIME}^\dagger = \text{TIME}^{-1} = -\text{TIME}. \quad (42)$$

As an hypernumber $\text{TIME} = -i_1$, minus the unique 2×2 matrix representing a 90 degrees rotation, the old commutative square root of -1 from which complex numbers appeared. The reason for considering this operator given

²⁷Remarkably, $-2i\text{TIME}$ is the hamiltonian operator of the damped quantum oscillator in the quantum theory of open systems; see N. Gisin and I. Percival, arXiv:quant-ph/9701024v1. In this theory based on the stochastic Schroedinger equation the role of torsion is central [62].

by the difference of nilpotents is because it plays the role of a comparison operator. Indeed, we have

$$\langle p|\text{TIME}|q \rangle = \bar{p}q - \bar{q}p = (1-p)q - (1-q)p = q - p = \bar{p} - \bar{q}. \quad (43)$$

TIME appears to be unchanged for unaltered states of consciousness:

$$\langle q|\text{TIME}|q \rangle = 0, \quad (44)$$

and if we have different cognitive states p, q , then $\langle p|\text{TIME}|q \rangle \neq 0$. So this operator does represent the appearance of a primitive difference on cognitive states, and it is antisymmetric and unitary. It is furthermore linked with a difference between annihilation and creation operators and thus stand for what we argued already as a most basic difference that leads to cognition and perception: the appearance of quantum jumps. Without them, no inhomogeneities nor events are accesible to consciousness, and even the very nature of self-reference as consciousness of consciousness requires such an operator for the joint constitution of the subject and the world. Thus its name, a TIME operator operator; it stands clearly in the subject side of the construction of a conception that overcomes the Cartesian gap, yet a subject that has superposed paradoxical states.

Let us consider next the eigenvalues of TIME, i.e. the numbers λ such that $\text{TIME}|q \rangle = \lambda|q \rangle$; they are obtained by solving the characteristic equation $\det|\text{TIME} - \lambda I| = \lambda^2 + 1 = 0$, so that they are $\lambda = \pm i$ with complex eigenstates

$$\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}. \quad (45)$$

They are not orthogonal, but self-orthogonal. We diagonalize TIME by taking

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \text{TIME} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (46)$$

so that

$$\text{TIME}_{\text{diag}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (47)$$

which as an hypernumbers we have that $\text{TIME}_{\text{diag}} = i_2$, so that $\text{TIME}_{\text{diag}}^2 = -I$. We want finally to comment that TIME is not a traditional clock, yet it allows to distinguish between after and before ($\rightarrow - \leftarrow$), forward and backwards. There is no absolute logical time, nor a privileged direction of it. To have a particular direction it must be asymmetrically balanced towards creation or annihilation. This can be computed as the complement of the operator phase²⁸

$$\overline{\cos(2a^\dagger) + \text{sen}(2a^\dagger)} = a^\dagger - a, \quad (48)$$

from which it follows that $\text{TIME} = \overleftarrow{-2} = \rightarrow - \leftarrow$, as we stated before.

Let us now retake M and decompose it as

$$M = \text{TIME} + \sigma, \text{ or still} \quad (49)$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (50)$$

The choice of σ for the second term is that it corresponds to $-\tilde{\sigma} = \bar{\sigma}$ as in eq. (17) above. Then we have that

$$\langle q|M|q \rangle = \langle q|\sigma|q \rangle. \quad (51)$$

Indeed, since $\langle q|\text{TIME}|q \rangle = 0$, so that the proof of eq. (51) follows. Furthermore we note that

$$\langle q|\sigma|q \rangle = \bar{q}^2 - q^2 = (\bar{q} - q)(\bar{q} + q) = \bar{q} - q. \quad (52)$$

from the normalization condition. Note here that the identity given by eq. (51) is a kind of quadratic metric in cognitive space which due to the normalization condition loses its quadratic character to become the difference in the cognitive values: $\bar{q} - q = 1 - 2q$ which becomes trivial in the undecided state in which $\bar{q} = q = \frac{1}{2}$.

The role of σ is that of a SPIN operator, as we shall name it henceforth, which coincides with the hypernumber ϵ_2 (or as a Pauli matrix is σ_z), so that $\sigma^2 = I$ the non-trivial square root of hypernumber $I = \epsilon_0$, which is the usual Pauli matrix σ_z in the decomposition of a Pauli spinor in the form

²⁸The complement of a logical operator L , is defined by $\bar{L} = I - L$.

$\sigma_x e_x + \sigma_y e_y + \sigma_z e_z$, for e_x, e_y, e_z the standard unit vectors in R^3 and we write their representations as hypernumbers

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \epsilon_3, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \epsilon_1, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \epsilon_2. \quad (53)$$

We can rewrite this average equation $\langle q|M|q \rangle = \langle q|\sigma|q \rangle$ as an average equation which the l.h.s. takes place in cognitive space of normalized states $|q \rangle$ and the r.h.s. in a Hilbert space of a two-state quantum system, say, spin-up $\psi(\uparrow)$, spin-down $\psi(\downarrow)$, so that the generic element is of the form $\psi = \psi(\uparrow)|0 \rangle + \psi(\downarrow)|1 \rangle$. Indeed, if we write $|q \rangle = \bar{\psi}(\uparrow)\psi(\uparrow)|0 \rangle + \bar{\psi}(\downarrow)\psi(\downarrow)|1 \rangle$, then the r.h.s. of eq. (51) is $\bar{q}^2 - q^2$, with $\bar{q} = \bar{\psi}(\uparrow)\psi(\downarrow)$, and $q = \bar{\psi}(\downarrow)\psi(\uparrow)$, so that equation (51) can be written as

$$\langle q|M|q \rangle = \langle \psi|\sigma|\psi \rangle \quad (54)$$

where the average of M is taken in cognitive states while that of the SPIN operator is taken in the two-state Hilbert space.

We revise the previous derivation for which the clue is the relation between cognitive states $|q \rangle$ and elements of two-state of Hilbert state $|\psi \rangle$ is that the former are derived from the latter by taking the complex square root of the latter, so that probability($|0 \rangle$) = $\bar{q} = \bar{\psi}(\uparrow)\psi(\uparrow)$ and probability($|1 \rangle$) = $q = \bar{\psi}(\downarrow)\psi(\downarrow)$, so that $\langle \psi|\sigma|\psi \rangle = \bar{q} - q = (\bar{q} - q)(\bar{q} + q) = \bar{q}^2 - q^2$. Therefore, by using the transformation between real cognitive states q defined by the complex square root of ψ , i.e. $q = \bar{\psi}\psi$, we have a transformation of the average of the cognitive operator M on cognitive states on the average of the spin operator on two-states quantum elements in Hilbert state, i.e. eq. (54). This is a very important relation, established by an average of the cognition operator (which transforms an orientable plane into a non-orientable Moebius surface due to the torsion introduced by M , as represented by eq.(25), and the spin operator on the Hilbert space of two-state quantum mechanics. It is an identity between the action of the cognizing self-referential mind and the quantum action of spin. Thus the cognitive logical processes of the subject become related with the physical field of spin on the quantum states. This is in sharp contrast with the Cartesian cut, and we remark again that this is due to the self-referential classical-quantum character of M as evidenced by eq. (25) which produces a torsion on the orientable cognitive plane of coordinates (true, false) to one which is torsioned to yield a superposed

state, S_- . The relation given by eq. (54) establishes a link between the operations of cognition and the quantum mechanical spin. This link is an interface between the in-formational and quantum realms, in which topology, torsion, logic and the quantum world operate jointly. Yet, due to fact that for the Klein bottle there is no inside nor outside, the exchange can go in both ways, i.e. the quantum realm can be incorporated into the classical cognitive dynamics, while the logical elements can take part in the quantum evolution. Indeed, if we have a matrix-logical string which contains the momentum product, say, $\dots \langle x|A|y \rangle \langle q|M|q \rangle \langle z|B|s \rangle \dots = \dots \langle x|A|y \rangle \langle \psi|\sigma|\psi \rangle \langle z|B|s \rangle \dots$. Thus, the factor $\langle \psi|\sigma|\psi \rangle$ entangles with the rest of the classical logical string creating a Schrodinger cat superposed state, since we have a string of valid propositions where one may be the negation of the other ²⁹

If we introduce the logical operators {NOT, AND, OR, TIME} and we think of them as variables (x, y, z, t) respectively, where NOT is the negation operator which we denote for short as N so that $N^2 = I$, AND is the conjunction \wedge and OR is the usual inclusive disjunction, \vee , so that

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \eta, \wedge = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \vee = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad (55)$$

then we have the following quantization of the 'spatial' logical operators by operator TIME ³⁰:

$$[\text{NOT}, \text{AND}] = [\text{NOT}, \text{OR}] = [\text{OR}, \text{AND}] = \text{TIME}. \quad (56)$$

²⁹This primordial role of spin as as protopsychic as well as protophysical is found also in the work, though not mathematically based, by Hu and Wu, who claim that spin is "the linchpin between mind and brain", though in a certain Cartesian way, associating spinor fields to processes in the brain and not to the processes of the mind; they further link it with self-referential processes alike the Klein bottle [25]. Spin and torsion, and its relations to time have been applied to qualitative analysis in economics and climatology in the works of Wu and Lin, and Lin [83]; the main difference with the present approach, is that the "yo-yo model" of these authors is based on first-order cybernetics instead of the second-order cybernetics of the present approach, so that their yo-yo is nothing else that the "inner" part of the Klein bottle.

³⁰Non-commutative spacetime was first advocated by Snyder already 60 years ago [73]. This was retaken in the work of Sidharth [72] in which it was further related to a Brownian motion related with Schroedinger equation with a backward and forward derivatives, alike in Nelson's stochastic mechanics [53] which in fact can be framed in terms of a geometry with trace-torsion which generates the random process or else is generated by it, in the work of the author [60,62].

Notice that in addition of TIME having an expression in terms of the creation and annihilation operators, $\text{TIME} = a - a^\dagger$, the difference between the annihilation and creation operators, furthermore, $a^\dagger + a = N$ and $a^\dagger a = \text{AND}$. Having already the product of annihilation and creation operators which are in fact nilpotent hypernumbers, then their multiplication allows to introduce any logical operator in terms of them. For example, $\text{OR} = a^\dagger + a + a^\dagger a$, $\text{YES} := I = a^\dagger a + a a^\dagger = \{a^\dagger, a\}$. It is remarkable that NOT can be derived being thus not primeval as in Aristotelian dualism: Indeed $[M, \text{AND}] = \text{NOT}$, and thus the set $\{M, \text{AND}\}$ form a complete functional basis. But this is not unique to this basis, also, say, since $[\text{TIME}, \text{AND}] = \text{NOT}$, it follows that also we have a complete functional basis with the set $\{\text{TIME}, \text{AND}\}$. While in classical logic the set $\{\text{AND}, \text{OR}\}$ is not functionally complete, in this framework this is the case since from the fact that $\text{TIME} = [\text{AND}, \text{OR}]$ as it can be easily verified, then we can obtain NOT from the identity $\text{NOT} = [\text{TIME}, \text{AND}]$ which together with the previous identity we obtain $\text{NOT} = [\text{AND}, [\text{AND}, \text{OR}]]$, so that now the set $\{\text{AND}, \text{OR}\}$ becomes functionally complete, i.e. all logical operators can be derived from them [76].

There is still another very remarkable role of these superposed states in producing a topological representation of a higher order form of self-reference, produced from oppositely twisted Moebius surfaces. So we shall consider the Cartesian modulo 2 sum of the superposed states

$$\mathcal{H} := |S_+ \rangle \oplus |S_- \rangle = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (57)$$

which we call the topological in-formation operator which is an hypernumber; indeed, $\mathcal{H} = \sigma_x + \sigma_z = \epsilon_3 + \epsilon_2$. We could have chosen the opposite direct sum or still place the minus sign on the first row in any of the columns and obtain a similar theory, but for non-hermitean operators unless the minus sign is on the first matrix element. Notice that it is a hermitean operator, which essentially represents the topological (or still, logo-topological) in-formation of a Klein bottle formed by two oppositely twisted Moebius surfaces. ³¹

³¹Alternatively we can introduce instead of \mathcal{H} another in-formation matrix for the Klein bottle, namely

$$\mathcal{H} := |S_+ \rangle \oplus |S_- \rangle = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (58)$$

which is non-hermitean.

The in-formation matrix satisfies $\mathcal{H}\mathcal{H}^\dagger = \mathcal{H}\mathcal{H}^{-1} = 2I$. We recognize in taking $1/\sqrt{2}\mathcal{H}$ the Hadamard gate in quantum computation [4], which due to the introduction of the $\frac{1}{\sqrt{2}}$ factor is hermitean and unitary. Now we have two orthogonal basis given by the sets $\{|0\rangle, |1\rangle\}$ and $\{|S_-\rangle, |S_+\rangle\}$ of classical and superposed states respectively, the latter un-normalized for which a factor $\frac{1}{2}$ has to be introduced but still does not give a unitary system as in quantum theory. An important role of the Klein bottle is precisely to transform these orthogonal basis, from classical states to superposed states which are nor classical nor quantum, but become quantized by appropriate normalization with the $\frac{1}{\sqrt{2}}$ factor. Indeed,

$$\mathcal{H}|0\rangle = |S_+\rangle, \mathcal{H}|1\rangle = |S_-\rangle, \quad (59)$$

and

$$\frac{1}{2}\mathcal{H}|S_+\rangle = |0\rangle, \frac{1}{2}\mathcal{H}|S_-\rangle = |1\rangle. \quad (60)$$

In the logical space coordinates (*true*, *false*) we have rotated the state $|0\rangle$ clockwise by 45 degrees through the action of \mathcal{H} and multiplied it its norm by 2, and for the state $|1\rangle$ we have rotated it likewise after being flipped. In reverse, the superposed states are transformed into the classical states by halving the in-formation matrix of the Klein bottle, producing 45 degrees counterclockwise rotations, one with a flip. Now classical and quantum states are functionally complete sets of eigenstates spanning each other. The classical states $|0\rangle$ and $|1\rangle$ can be easily determined to be the eigenstates of AND, and the superposed states $|S_-\rangle$, $|S_+\rangle$ are the eigenstates of NOT. It is known that the logical basis of operators {AND, OR} is functionally complete, generating all operators. Hence our system of classical and superposed (or still, quantum by appropriate normalization by $\frac{1}{\sqrt{2}}$) eigenstates constitute together a functionally complete system. This system is self-referential. Furthermore, there are operators which produce the rotation of one orthogonal system on the other orthogonal system. The logical differentiation operator M defined by the commutator [FALSE, TRUE] or still eq. (25) transforms classical states $|x\rangle = \bar{x}|0\rangle + x|1\rangle$ into $|S_-\rangle$ and still the anticommutator {FALSE, TRUE} which coincides with the matrix $\mathbf{1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ transforms $|x\rangle$ into $|S_+\rangle$, i.e.

$$M|x\rangle = |S_-\rangle, \mathbf{1}|x\rangle = |S_+\rangle. \quad (61)$$

which can be rephrased by saying that M evidences on its action on a classical state the torsion in the quantum commutator of FALSE and TRUE while the ONE operator $\mathbf{1}$ transforms $|1\rangle$ into $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |S_-\rangle$. Since both M and $\mathbf{1}$ are non-invertible, we shall use instead the fact that $\mathcal{H}^{-1} = \frac{1}{2}\mathcal{H}$, so that in addition of the transformation by the Klein bottle of the classical basis in eq. (59), the reversed transformation from the superposed to the classical states is achieved by

$$\frac{1}{2}\mathcal{H}|S_+\rangle = |0\rangle, \quad \frac{1}{2}\mathcal{H}|S_-\rangle = |1\rangle. \quad (62)$$

Yet, we stress again that these transformations are not unitary which is easily resolved by the $\frac{1}{\sqrt{2}}$ factor and then we have a transformation of classical into quantum states and viceversa. In the latter case, the renormalized Klein bottle acts like a quantum operator producing coherent quantum states, a topological Schroedinger cat state which does not decohere.

4.3 The logical quaternions

We wish to construct in this setting the logical equivalent of the quaternions following a related idea of their construction using the iterants in the calculus of indications by Kauffman [38], $\{1, \vec{i}, \vec{j}, \vec{k}\}$, with $\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$, $\vec{i}\vec{j}\vec{k} = -1$. We start by the scalar 1 given by the hypernumber $1 = \epsilon_0$, i.e. the identity matrix I . We define $\vec{i} := \sigma N$, where σ and N are the spin and negation operator respectively. To define both \vec{j} and \vec{k} we take an additional square root of -1 , namely the hypernumber $i_2 = \text{TIME}_{\text{diag}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, to define next $\vec{j} = -i_2\sigma$ so that $\vec{j} = -iI$ and finally $\vec{k} := i_2N$ so that $\vec{k} = i\text{TIME}$ which as a Pauli matrix is σ_y . This system has the fundamental properties of the quaternions. Notice that we have needed in addition of I, N and σ , the hypernumber $\text{TIME}_{\text{diag}}$ to be able to introduce \vec{j} and \vec{k} . So, to introduce the logical quaternions, both components, SPIN and TIME of the logical momentum are needed, the latter diagonalized. This is quite remarkable, since these operators are implicit to the construction of this four-dimensional spacetime in which TIME is not a fourth vector.

4.4 Electrons, Neutrinos and W-Bosons

We here present in the framework of matrix logics an idea due to Kauffman developed in the framework of Spencer-Brown's laws of form [39]. If we choose the neutrino v to be represented by $\text{NOR} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the antineutrino, $\bar{v} = \wedge = \text{AND}$, then $v\bar{v} = \mathbf{0}$. Now for the electron e and the W^- -boson, we take $\bar{v}N = \wedge N$ so that $e = W^- = a^\dagger$. Finally, for W^+ we take $vN = \text{NOR.NOT}$ so that $W^+ = a$. Then

$$W^-e = (a^\dagger)^2 = \mathbf{0}, \quad (63)$$

$$W^-v = a^\dagger v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}, \quad (64)$$

$$W^+e = aa^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = v, \quad (65)$$

$$W^+v = av = \mathbf{0}. \quad (66)$$

5 Matrix Logic and the Calculus of Indications

Inasmuch the calculus of indications integrates operators and operands, we want to show how this also appears in matrix logic. Furthermore, we shall show that matrix logic is generated from the solutions of the paradoxical equation in the Boolean interpretation of the calculus of indications. So we start with the linearly independent basis produced by the tensor product in matrix logic of the Boolean states,

$$|0 \rangle \langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, |0 \rangle \langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (67)$$

$$|1 \rangle \langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, |1 \rangle \langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (68)$$

which in terms of the paradoxical waveforms \mathcal{I} and \mathcal{J} of the calculus of indications can be written as

$$|0 \rangle \langle 0| = \mathcal{I}, \quad |0 \rangle \langle 1| = \mathcal{I}N, \quad |1 \rangle \langle 0| = \mathcal{J}, \quad |1 \rangle \langle 1| = \mathcal{J}N. \quad (69)$$

where instead of using η we have written its matrix logic operator notation, N , the negation. From this tensor basis we can now generate all operators. For instance (the second one for N as well as the fourth for SPIN, being trivial identities)

$$\wedge = \text{AND} = |1 \rangle \langle 1|, \quad (70)$$

$$N = |1 \rangle \langle 0| + |0 \rangle \langle 1| = \mathcal{J} + \mathcal{I}N, \quad (71)$$

$$\text{OR} = \vee = |1 \rangle \langle 0| + |0 \rangle \langle 1| + |1 \rangle \langle 1| = N + \wedge, \quad (72)$$

$$\text{SPIN} = \sigma = -|1 \rangle \langle 1| + |0 \rangle \langle 0| = -[0, 1] + [1, 0] = [1, -1], \quad (73)$$

$$\begin{aligned} \text{TIME} &= |0 \rangle \langle 1| - |1 \rangle \langle 0| = [1, 0]N - [0, 1]N \\ &= [1, -1]N = \sigma N, \end{aligned} \quad (74)$$

so that TIME is indeed minus the operator defined in eq. (17). Since $\{\text{TIME}, \text{AND}\}$ and also $\{\text{AND}, \text{OR}\}$ are both functionally complete basis, i.e. we can generate all the operators from them, and ultimately from the wavepatterns $\{\mathcal{I}N, \mathcal{J}, \mathcal{J}N\}$ for both cases, which requires the imaginary wavepatterns \mathcal{I}, \mathcal{J} and N which as we know is the original distinction () in the Boolean interpretation of the calculus of distinctions. In other words, we need the Boolean duality operator given by N and both the imaginary waveforms that are solutions of the paradoxical equation $f = (\bar{f})$, i.e. topologically we need a boundary and the Klein bottle. In this sense, it has appeared most surprisingly that matrix logic is a connective-free logical calculus based on a surface of paradox, the Klein bottle and a primitive distinction, which we can think of as having been stretched to higher dimension to reenter itself completing the holomovement and thus complete the first period of the imaginary solutions and thus infinitely recurring. At the level of matrix logic this translates into the basic tensor product of the Boolean states so all that we need in this framework is a pair of adjoint spaces, those of bras and kets. What is remarkable of this is the fact that there has been a complete bouleversement of logic as was understood, since it is clear from this that the solutions to the paradoxical equation $f = (\bar{f})$ in the calculus of indications, as imaginary solutions that can be either be seen as patterns on space of ever increasing depth and irresolvable to simplification, or as time waves, appear as generating all the operators of matrix logic in terms of the basis of the tensor product. The fact that these imaginary solutions can be framed in a Boolean setting as proved by Kauffman and Flagg [38], does not counter the self-referential and multivalued matrix logic that arouses from the calculus of

distinctions and the Klein bottle but adds an additional flavour to the complexity of the issues. This new perspective of the generation of a complex thinking space from the calculus of distinctions is most remarkable, in account that quantum and fuzzy logics are particular cases of matrix logic [77]. Furthermore, it places the calculus of distinctions in precedence of quantum mechanics, as is the case already of matrix logic as understood by its founder, August Stern [76,77].

6 Conclusions

This article has presented a mathematical, physical, cognitive, perceptual, logical, cybernetic and semiotic conception in which the Cartesian cut has been overcome. The 'outside' world has been turned 'inside' and as the real, imaginary, symbolic and time realms have appeared as unseparable, just like it was proposed in the work of Lacan [42]. We would like to venture some ideas for closing remarks to this conception. The mind-matter problem as usually stated starts by the Cartesian cut, separates the mind from the brain; for neurological evidence in the contrary and the relevance of higher order cognitive functions such as emotions and feelings we refer to Damasio [10]. In this problem, the brain stands in the object side of the cut and since the designative value of Aristotelian logic points to the material objective world, the mind (as the subject qua subject) is cast away and it has to be brought back somehow into this mapping. So, to many researchers consciousness cannot but be treated but as an epiphenomenon, an emergent property of the material world. Thus electric current in the neurons have been invoked in the Hodgkin-Huxley model as the first physical basis [24], and more contemporarily with the work of Penrose and Hammeroff, quantum effects in microtubules have been proposed together with non-algorithmic properties which are related to the incompleteness of the Goedel theorem, to explain the working of the mind and the collapse of the wave function [56]. In these approaches, which are mechanical either classical or quantum, self-reference is completely absent, and as we have already seen, by proceeding likewise, the holographic integration of subject with object into a higher-order structure and functionality is neglected completely. Consequently, time and logic will be required to be imposed to the Cartesian models as exterior ad-hoc structures to which additional Cartesian models will be required to integrate

them; in some conceptions they will be conceived as a Platonic world, yet one in which the subject is cast as passive. In this article we have seen that self-reference, topology, time, logic and geometry both of the cognitive and spacetime structures, the classical and quantum worlds, are interwoven in the lifeworld of fused subject-with-object (with the proper qualifications already presented), so that in the Cartesian cut approach implicit to the mind-matter problem, they would be required to be introduced as exterior to the subject in the classical formula we have already discussed at the beginning of this article. With respect to the calculus of distinctions which has departed from the original distinction -which we have conceived as a semiotic codification of a torsion field which thus has become a logo-physical-geometrical field-together with the imaginary solutions of the paradoxical equation, we have seen that it renders a 4-state logic and furthermore the matrix logic due to Stern; the algebra related to the former logic is associated to a complete algebra that we described in Section 3.2, so that the Goedel theorem has no bearing to the present conception [66]. While as we have seen that already the visual organization and functioning of the neurocortex has the Klein bottle for topology (which is further related to holography after D. Gabor [44]), acts as an in-formation matrix that transforms classical states into quantum ones, and viceversa; hence, it is clear then that this does not rule out the presence of quantum physics at the level of the brain and the mind, which in the Klein bottle conception are transformable one into the other. This exchange of mind and brain, for two-state quantum systems is given by the eq. (54) and in the sense we discussed above, are in a sense exchangeable. Hence, 'classical' and 'quantum' may seem to be, to some extent, labels that are expendible, would not be that the former states can be obtained as the final result of a string of logical operators actions that may be mediated by superposed states. This is the 'collapse' of the quantum system yet with no need of the collapse of the wave-function itself in distinction to the usual approach as in Penrose [56]), or we can have alternatively a string of classical states that can be transformed with the final action of the (renormalized) Klein bottle to a superposed quantum state, leaving out of this conception a clear-cut separation of the classical and the quantum world. Again, this is the very structure of the Klein bottle, not an artifice of the 'observer' whose mind wavefunction is demanded to collapse or be collapsed by the system in a Cartesian mindset in some interaction. We have also found that the Klein bottle is -but not quite- the implicate order that

Bohm was searching for [7], yet in an unexpected way, since the hole (which in the Cartesian cut and in Bohm as well, is related to the explicate order as a particle-singularity) generates the whole structure of the Klein bottle (the implicate order) which will return to the hole. It is clear then that there is an integrative meta-algorithmic level in which Bohm's implicate and explicate order are instances which are transformable between themselves; for a remarkable elaboration in a different approach on Bohm's conception we refer to Pylkkanen [7]. This meta-algorithmic level, first proposed by Johansen and remarkably then unrelated to the Klein bottle [35], is the one in which time, self-reference, signs, language, space, the processes of the mind, emotions, feelings, are interwoven in an unseparable way, rendering the Cartesian cut a disjointed analytical dissection which leaves out much of the richness of this tapestry of which we are woven into and weave ourselves by the very act of being, of which thinking is no epiphenomenon, but woven into this meta-lifeworld. In this, language (in its manifold renderings) is the semiotic lifeworld of the subject-blended-into-object, so unless we would reintroduce the Cartesian cut, we can say that the world indicates itSelf through ourselves which by doing this we indicate ourselves and itSelf, and as in Escher's pictures, there is a synthesis of this which is the meta-algorithmic lifeworld.

Some questions are in order -let us restrict ourselves to physics- in this next step in which we shall turn in the Klein bottle to face the light of the 'outside' and interrogate us where to place/find the time operator that we have encountered in the decomposition of the cognition operator in terms of it and a spin operator which in average translates cognitive spin to physical spin and viceversa, or still in the difference between the implication and converse implication operators, as the difference between two nilpotent hypernumbers. A particular treat of hypernumbers is the fact that they admit square roots, and still a most rich geometrical interpretation which here we have not discussed. So where are we to find in the 'outer' world the correlate of this 'internal' time? Is it in the Newtonian absolute time, or is it in the universal time parameter of quantum field theory encountered first by Stueckelberg [82] and further studied by Horwitz and Piron [27], and Fanchi [13]? But these are one-dimensional variables and not operators, so we can further enquire on the relation between the time operator of cognitive states and its possible relations with the time operators discovered by Antoniou [3], and separately, Horwitz [29]. How does the present topological-geometrical-logical model relate to the 'inner' time projective geometry that Saniga has

found in 'altered' states of consciousness [69]?. In principle, the very generation of a normalized cognitive state $|q\rangle$ is related to a projective structure, so it would be natural to try to establish if there is a natural relation between the two lines of research. Since spin and cognition appear related in two-state systems for real Hilbert spaces, it would be very interesting to attempt to extend this to octonionic quantum mechanics as developed by Horwitz [28] or further to an hypernumber quantum mechanics in account of the fundamental role that non-trivial square roots of $+1$ have appeared to play in the present approach.

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